**Solution of the Third HKUST Undergraduate Math Competition – Senior Level**

1. (*Solution 1 is a common proof involving* $\sqrt{3}$) Assume $A, B, C$ are lattice points and also the vertices of an equilateral triangle. By translation, we may assume $A = (x_0, y_0), B = (x_1, y_1), C = (0, 0)$. Let $r = \sqrt{x_0^2 + y_0^2} = \sqrt{x_1^2 + y_1^2}$. Interchanging the names of $A$ and $B$ if necssary, we may suppose $x_0 = r\cos\theta, y_0 = r\sin\theta$ and

$$x_1 = r\cos(\theta + \pi/3) = (x_0 - y_0\sqrt{3})/2 \quad \text{and} \quad y_1 = r\sin(\theta + \pi/3) = (y_0 + x_0\sqrt{3})/2.$$

One of $x_0$ or $y_0$ is nonzero. If $x_0 \neq 0$, then $(2y_1 - y_0)/x_0 = \sqrt{3}$, which is a contradiction. If $y_0 \neq 0$, then $(x_0 - 2x_1)/y_0 = \sqrt{3}$, which is also a contradiction.

(*Solution 2 is a proof without any* $\sqrt{3}$ *due to HUANG Yifeng*) If $A, B, C$ are lattice points that are vertices of an equilateral triangle, then we may assume $C = (0, 0)$ without loss of generality (just replace $A, B$ by $A - C, B - C$). Then we have $\|A\| = \|B\|$ and

$$\frac{A \cdot B}{\|A\|\|B\|} = \cos 60^\circ = \frac{1}{2}.$$

So $2A \cdot B = \|A\|^2 = \|B\|^2$. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Removing common factor in $\mathbb{Z}$, we may assume $\gcd(x_1, y_1, x_2, y_2) = 1$.

Now we claim all $x_1, y_1, x_2, y_2$ are odd. If one is even, say $x_1$, then $x_1^2 + y_1^2 = 2(x_1x_2 + y_1y_2) = x_2^2 + y_2^2$ implies $y_1$ is even. So $x_1^2 + y_1^2 = x_2^2 + y_2^2$ is divisible by 4. Hence, $x_2, y_2$ are even because otherwise $x_2^2 + y_2^2 \equiv 1$ or $2 \pmod 4$. This contradicts $\gcd(x_1, y_1, x_2, y_2) = 1$, which establishes the claim. Then $2(x_1x_2 + y_1y_2) \equiv 2(1 + 1) \equiv 0 (\text{mod } 4)$, contradicting $x_1^2 + y_1^2 \equiv 1 + 1 = 2 (\text{mod } 4)$.

2. (*Solution 1*) If $x_1, x_2, x_3, \ldots$ converges to $x$, then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x)$. Now $x = f(x)$ if and only if $x^3 - 2x^2 + x - 1 = 0$. Using calculus, we can study the graph of the function $p(x) = x^3 - 2x^2 + x - 1$. We can see $p(x)$ has exactly one real root $r$, which is in the interval $(1, 2)$.

Now for every $x_1 = a \in \mathbb{R}$, $x_n \in f(\mathbb{R}) = [1, 2)$ for $n = 2, 3, 4, \ldots$. Observe that on $[1, 2)$, we have $x \leq f(x)$ iff $x(1+x^2) \leq 1+2x^2$ iff $p(x) \leq 0$ iff $x \leq r$. Also, since $f'(x) = 2x/(1+x^2)^2 > 0$ on $[1, 2)$, $x \leq r$ implies $f(x) \leq f(r) = r$. If $x_2 < r$, then $x_2, x_3, x_4, \ldots$ is increasing and bounded above by $r$. Similarly, if $x_2 \geq r$, then $x_2, x_3, x_4, \ldots$ is decreasing and bounded below by $r$. Hence, $x_1, x_2, x_3, \ldots$ converges to some $x \in [1, 2]$. By the last paragraph, $x = r$ for every $a \in \mathbb{R}$.

(*Solution 2 based on the work of PANG Lok Wing*) The function $f : [1, 2] \to [1, 2]$ with $f(x)=1 + \dfrac{x^2}{1 + x^2}$ is a contractive mapping because

$$|f(x) - f(y)| = \left|\frac{x^2}{1 + x^2} - \frac{y^2}{1 + y^2}\right| = \left|\frac{(x + y)(x - y)}{1 + x^2 + y^2 + x^2y^2}\right| \leq M|x - y|,$$

where $M = \max\left\{\dfrac{x + y}{1 + x^2 + y^2 + x^2y^2} : (x, y) \in [1, 2] \times [1, 2]\right\} = \dfrac{x_0 + y_0}{1 + x_0^2 + y_0^2 + x_0^2y_0^2} < 1$ for some $(x_0, y_0)$ in $[1, 2] \times [1, 2]$. By the contractive mapping theorem, $f$ has a unique fixed point $r$ and every sequence $x_1, x_2, x_3, \ldots$ satisfying $x_1 = a \in \mathbb{R}, x_2 \in [1, 2]$ and $x_{n+1} = f(x_n)$ converges to $r$.

3. Since $A$ is positive definite, there exists an orthogonal matrix $U$ such that $A = UEU^{-1}$, where $E = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ is a diagonal matrix with all $d_i$'s positive. Take $B = U \operatorname{diag}(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n})U^{-1}$. Then $B$ is symmetric and $A = B^2$.

For $C$ and $D$, by the last paragraph, $C = F^2$ for some symmetric matrix $F$ and $CD = F^2D = F(FDF)F^{-1}$. So $CD$ and $FDF$ have the same eigenvalues. However, $FDF$ is positive definite, since for every $x \in \mathbb{R}^n \setminus \{0\}$, $x^TFDFx = (Fx)^TD(Fx) > 0$. So all the eigenvalues of $FDF$ are real and positive.

4. Clearly $a \le b$. Assume $a < b$. let $p$ be a prime number greater than $b$. Let $n = (a+1)(p-1)+1$. By Fermat's little theorem, $a^n \equiv a (\text{mod } p)$ and $b^n \equiv b (\text{mod } p)$. Now $n \equiv -a (\text{mod } p)$. Then $a^n + n \equiv a + n \equiv 0 (\text{mod } p)$. So $b^n + n \equiv 0 (\text{mod } p)$. However, $b^n + n \equiv b - a (\text{mod } p)$ and $0 < b - a < b < p$, contradiction. So $a = b$.

5. Let $F(y) = \displaystyle\int_y^{y+b-a} f(x)\, dx$. Then $F(a) = \alpha$. By the fundamental theorem of calculus, $F'(a) = f(b) - f(a)$. It suffices to show $F'(a) = 0$. Since $f$ is continuous and nonnegative on $\mathbb{R}$, $F(y)$ is continuous, nonnegative and tends to 0 as $y \to \pm\infty$. So $F(y)$ must attain a maximum value at some point.

   Assume $F(a)$ is not the maximum value of $F$. Then there exists $y_0$ such that $F(y_0) > F(a)$. Let $G(y) = \displaystyle\int_{y_0}^{y} f(x)\, dx$. Then $G(y_0) = 0$ and $G(y_0 + b - a) = F(y_0) > F(a)$. Since $f$ is continuous, so is $G$. The intermediate value theorem implies that for some $y_1 \in (y_0, y_0 + b - a)$, we have $G(y_1) = F(a) = \alpha$. Since $[y_0, y_1]$ has shorter length than $[a, b]$, this contradicts the condition that $[a, b]$ is an interval of minimal length such that $\displaystyle\int_b^a f(x) = \alpha$. Therefore, $F(a)$ is the maximum value of $F$ and $f(b) - f(a) = F'(a) = 0$.

6. By Schwarz' lemma, $f_1(z) = f(z)/z$ satisfies $|f_1(z)| \le 1$ for all $z \in D$. The function $g(z) = (z-r)/(1-rz)$ is a bijective analytic map from $D$ to $D$. Applying Schwarz' lemma to $f_2(z) = f_1\Big(\dfrac{z-r}{1-rz}\Big)$, we see that $f_3(z) = f_1(z)/\Big(\dfrac{z+r}{1+rz}\Big)$ satisfy $|f_3(z)| \le 1$ for all $z \in D$. Similarly, applying Schwarz' lemma to $f_4(z) = f_3\Big(\dfrac{z+r}{1+rz}\Big)$, we see that $f_5(z) = f_3(z)/\Big(\dfrac{z-r}{1-rz}\Big)$ satisfy $|f_5(z)| \le 1$ for all $z \in D$. Then

$$|f(z)| \le |z| \Big|\frac{z-r}{1-rz}\Big| \Big|\frac{z+r}{1+rz}\Big| |f_5(z)| \le |z| \Big|\frac{z-r}{1-rz}\Big| \Big|\frac{z+r}{1+rz}\Big|.$$