Fourth HKUST Undergraduate Math Competition – Junior Level

April 30, 2016

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1. For every positive integer n, let $f(n)$ be the number of ordered pairs (x, y) of positive integers satisfying $n(x + y) = xy$.

(a) Prove that for every positive integer n, $f(n)$ is odd.

(b) Find, with proof, a two-term formula for $f(2^n)$.

Problem 2. Determine all continuous functions $f : (0, +\infty) \to \mathbb{R}$ such that $f(1) = 3$ and for all $x, y > 0$,

$$\int_1^{xy} f(t)\ dt = x \int_1^y f(t)\ dt + y \int_1^x f(t)\ dt.$$

Problem 3. Let $H_n = \sum_{k=1}^n 1/k$. Show that the infinite series

$$\sum_{n=1}^{\infty} \frac{H_{n+1}}{n(n+1)}$$

converges and find its value.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and $f''(x) \leq 0$ for all $x \in \mathbb{R}$. Prove that

$$\int_0^1 f(t^2)\ dt \leq f\Big(\frac{1}{3}\Big).$$

Problem 5. Let A and B be $n \times n$ matrices over $\mathbb{R}$ such that $A + B$ is invertible. Prove that $A(A + B)^{-1}B = B(A + B)^{-1}A$.

Problem 6. Let $p(z)$ be a nonconstant polynomial with real coefficients and only real roots. Prove that for every real number r, the polynomial $p(z) - rp'(z)$ has only real roots.

– End of Paper –