Solution of the Fourth HKUST Undergraduate Math Competition – Junior Level

1. (a) Note that the solutions (x, y) with $x \neq y$ come in pairs since (x, y) is a solution if and only if (y, x) is a solution. Thus, there are an even number of such solutions. On the other hand, there is exactly one solution with $x = y$, namely $x = 2n$. Therefore the total number of solutions, $f(n)$, is odd.

   (b) Rewrite the equation $n(x + y) = xy$ as (*) $(x - n)(y - n) = n^2$. The two factors $x - n$ and $y - n$ must be either both positive or both negative. In the latter case, we have $1 \le x < n$ and $1 \le y < n$, so $(x - n)(y - n) = (n - x)(n - y) < n^2$, which contradicts (*). Thus we necessarily have $x - n \ge 1$ and $y - n \ge 1$. Setting $a = x - n$ and $b = y - n$, the number, $f(n)$, of solutions to (*) is seen to be equal to the number representations $n^2 = ab$ with $1 \le a, b \le n^2$. But the latter is equal to the number of divisors of n^2. In particular, since 2^{2n} has exactly $2n + 1$ positive divisors (namely, $2^0, 2^1, \ldots, 2^{2n}$), we have $f(2^n) = 2n + 1$.

2. Let $y = 1$. Then $xf(x) = 3x+g(x)$, where $g(x) = \int_1^x f(t)\,dt$. Then $xg'(x)-g(x) = 3x$ and $g'(1) = 3$. Using integrating factor $1/x^2$, we have $(g(x)/x)' = 3/x$. Then $g(x) = x(3\ln x+C)$. So $f(x) = g'(x) = 3\ln x+3$.

3. For any positive integer N we have

$$\sum_{n=1}^{N} \frac{H_{n+1}}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{H_{n+1}}{n} - \frac{H_{n+1}}{n+1} \right) = \sum_{n=1}^{N} \left(\frac{H_n + 1/(n+1)}{n} - \frac{H_{n+1}}{n+1} \right)$$
$$= \sum_{n=1}^{N} \left(\frac{H_n + 1}{n} - \frac{H_{n+1}+1}{n+1} \right) = \frac{H_1+1}{1} - \frac{H_{N+1}+1}{N+1} = 2 - \frac{H_{N+1}+1}{N+1}.$$

   Now,
$$\frac{H_{N+1}}{N+1} = \frac{1}{N+1}\sum_{k=1}^{N+1}\frac{1}{k} \le \frac{1}{N+1} + \frac{1}{N+1}\int_1^{N+1}\frac{1}{x}\,dx = \frac{1}{N+1} + \frac{\log(N+1)}{N+1},$$
   which goes to 0 as $N \to \infty$. Hence the partial sums converge to the value 2 as $N \to \infty$.

4. By Taylor's theorem, for all $x \in [0, 1]$,
$$f(x) = f\big(\tfrac{1}{3}\big) + f'\big(\tfrac{1}{3}\big)\big(x - \tfrac{1}{3}\big) + \frac{f''(\theta_x)}{2}\big(x - \tfrac{1}{3}\big)^2 \le f\big(\tfrac{1}{3}\big) + f'\big(\tfrac{1}{3}\big)\big(x - \tfrac{1}{3}\big).$$
   Since $x^2 \in [0, 1]$, we have $f(x^2) \le f\big(\tfrac{1}{3}\big) + f'\big(\tfrac{1}{3}\big)\big(x^2 - \tfrac{1}{3}\big)$. So
$$\int_0^1 f(x^2)\,dx \le f\big(\tfrac{1}{3}\big) + f'\big(\tfrac{1}{3}\big)\int_0^1 \big(x^2 - \tfrac{1}{3}\big)\,dx = f\big(\tfrac{1}{3}\big).$$

5. $A(A+B)^{-1}B = A(A+B)^{-1}[(A+B)-A] = A(A+B)^{-1}(A+B) - A(A+B)^{-1}A = A - A(A+B)^{-1}A = A - [(A+B) - B](A+B)^{-1}A = A - A + B(A+B)^{-1}A = B(A+B)^{-1}A$.

6. The case $r = 0$ is clear. For $r \neq 0$, let n be the degree of $p(z)$ and $x_1 < x_2 < \cdots < x_k$ be the roots of $p(z)$ with multiplicities $m_1, m_2, \ldots, m_k$, respectively. If any m_j exceeds 1, then $p(z) - rp'(z)$ has a root at x_j with multiplicities $m_j - 1$ (giving a total of $n - k$ roots all together). We have $p(z) = c(z - x_{k_1})^{m_1}(z - x_{k_2})^{m_2}\cdots(z - x_{k_k})^{m_k}$. Then $\dfrac{p'(z)}{p(z)} = \displaystyle\sum_{j=1}^{k}\frac{m_j}{z - x_j}$. Since $\displaystyle\lim_{x\to x_j^-}\frac{p'(x)}{p(x)} = -\infty$ and $\displaystyle\lim_{x\to x_j^+}\frac{p'(x)}{p(x)} = +\infty$, by the intermediate value theorem, there exists $w_j \in (x_j, x_{j+1})$ such that $p'(w_j)/p(w_j) = 1/r$, Then the w_j's are also roots of $p(z) - rp'(z)$. The x_j's and w_j's account for $(n - k) + (k - 1) = n - 1$ roots of $p(z) - rp'(z)$. The last root must also be real as nonreal complex roots come in conjugate pairs. So $p(z) - rp'(z)$ has only real roots.