Fourth HKUST Undergraduate Math Competition – Senior Level

April 30, 2016

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1 Let V be a real vector space and $T : V \to V$ be a linear transformation. If $v_1, v_2, \ldots, v_n$ are non-zero eigenvectors of T with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively such that $\lambda_i \neq \lambda_j$ for $i \neq j$, then prove that $v_1, v_2, \ldots, v_n$ are linearly independent.

Let $c_1, c_2, \ldots, c_n$ be positive numbers such that $c_1 < c_2 < \cdots < c_n$. Show the functions $f_i(x) = \sin(c_i x)$ $(i = 1, 2, \ldots, n)$ are linearly independent real-valued functions on the real line.

Problem 2. Given a set $S = \{a_1, a_2, \ldots, a_k\}$ with $a_1 > a_2 > \ldots > a_k$, define its alternating sum by $A(S) = a_1 - a_2 + a_3 - \cdots + (-1)^{k+1} a_k$. For example, $A(\{4\}) = 4$, $A(\{7, 3, 1\}) = 7 - 3 + 1 = 5$. Find, with proof, a one-term formula for the sum of $A(S)$ over all non-empty subsets S of $\{1, 2, \ldots, n\}$.

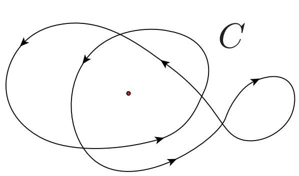
Problem 3. Let i be the square root of -1 in $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Let $\mathrm{SL}_2(\mathbb{R})$ be the group of 2×2 real matrices with determinant 1. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $z \in H$, define action $g \cdot z = \dfrac{az + b}{cz + d}$.

(1) Prove that $\{g \in \mathrm{SL}_2(\mathbb{R}) \mid g \cdot i = i\} = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$ and for every $z \in H$, there exists $g \in \mathrm{SL}_2(\mathbb{R})$ such that $g \cdot i = z$.

(2) Prove that for any $z, w \in H$, there exists $g \in \mathrm{SL}_2(\mathbb{R})$ such that $g \cdot z$ and $g \cdot w$ are both on the vertical line passing through i.

Problem 4. Let $X_n = \{1, 2, \ldots, n\}$ and Y_n be the set of all ordered pairs (A, B), where A and B are non-empty disjoint subsets of X_n. Prove the number of elements in Y_n is $3^n - 2^{n+1} + 1$.

Problem 5. Show that the power series representation of $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n (z-1)^{2n}}{n!}$ with center at 0 cannot have three consecutive zero coefficients.



Problem 6. Calculate the line integral $\displaystyle\oint_C \frac{y\,dx - x\,dy}{4x^2 + y^2}$ over the closed path C in $\mathbb{R}^2$ (as shown on the left), where the dot represents the origin.

– End of Paper –