Solution of the Fourth HKUST Undergraduate Math Competition – Senior Level

1. Assume there exists a linear dependent relation $a_1v_{n_1}+\cdots+a_kv_{n_k}=0$ with all $a_i\neq 0$ and k minimal. Note $k>1$. We have $a_1\lambda_{n_1}v_{n_1}+\cdots+a_k\lambda_{n_k}v_{n_k}=T(a_1v_{n_1}+\cdots+a_kv_{n_k})=0$. Since $k>1$, not all $\lambda_{n_i}=0$, say $\lambda_{n_k}\neq 0$. Then $a_1(\lambda_{n_k}-\lambda_{n_1})v_{n_1}+\cdots+a_1(\lambda_{n_k}-\lambda_{n_{k-1}})v_{n_{k-1}}=0$ with all $a_i(\lambda_{n_k}-\lambda_{n_i})\neq 0$ $(i=1,\ldots,k-1)$. This contradicts the minimality of k. So $v_1,\ldots,v_n$ are linearly independent.

   Next, let V be the vector subspace spanned by $\sin(c_1x),\ldots,\sin(c_nx)$ in the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let T be the linear transformation d^2/dx^2. Then the $-c_i^2$'s are distinct eigenvalues of T with distinct $\sin(c_ix)$ as eigenvectors. By the first part of the problem, $\sin(c_1x),\ldots,\sin(c_nx)$ are linearly independent.

2. Let $f(n)$ denote the above sum. For convenience we define $A(\emptyset)=0$ and include the empty set in the above summation. We will show that (*) $f(n)=n2^{n-1}$.

   For $n=1$, the sum reduces to $A(\{1\})=1$, which proves (*) in this case. Now assume $n\geq 2$. The subsets $S\subseteq\{1,2,\ldots,n\}$ are either of the form $S=S'$, where $S'\subseteq\{1,2,\ldots,n-1\}$, or of the form $S'\cup\{n\}$, where $S'\subseteq\{1,2,\ldots,n-1\}$. In the latter case, we have $A(S)=A(S'\cup\{n\})=n-A(S')$. Hence,

$$f(n)=\sum_{S'\subseteq\{1,2,\ldots,n-1\}}A(S')+\sum_{S'\subseteq\{1,2,\ldots,n-1\}}(n-A(S'))=\sum_{S'\subseteq\{1,2,\ldots,n-1\}}n=n2^{n-1},$$

   since there are 2^{n-1} subsets S' of $\{1,2,\ldots,n-1\}$. This proves (*) for $n\geq 2$.

3. (1) Let $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. We have $g\cdot i=\dfrac{ai+b}{ci+d}=i$ iff $ai+b=(ci+d)i$ iff $a=d, b=-c$ iff $g=\begin{pmatrix}a & b\\ -b & a\end{pmatrix}$. Since $\det g=1$, so $a^2+b^2=1$. So g has the form $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$. For $z\in H$, we write $z=b+ai$, $b\in\mathbb{R}$, $a>0$, then

$$\begin{pmatrix}a^{1/2} & a^{-1/2}b\\ 0 & a^{-1/2}\end{pmatrix}\cdot i=b+ai=z.$$

   (2) From (1), we may assume that $z=i$. If i and w are on the upper imaginary axis, then take g to be the identity matrix I. Otherwise, $\operatorname{Re}w\neq 0$. We claim we can find $g=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\neq -I$ such that $g\cdot w$ is on the upper imaginary axis. Then $g\cdot i=i$ and $g\cdot w$ are both on the upper imaginary axis. Observe

$$\operatorname{Re}g\cdot w=\operatorname{Re}\left(\frac{\cos\theta w+\sin\theta}{-\sin\theta w+\cos\theta}\right)=\frac{(-\tan^2\theta\operatorname{Re}w+\tan\theta(1-|w|^2)+\operatorname{Re}w)\cos^2\theta}{|\underbrace{1-\sin\theta w+\cos\theta}_{\neq 0\ as\ w\in H,\ g\neq -I}|^2}.$$

   Now the equation $-x^2\operatorname{Re}w+x(1-|w|^2)+\operatorname{Re}w=0$ has a nonzero real root $x=\alpha$ since the discriminant is $(1-|w|^2)^2+4(\operatorname{Re}w)^2\geq 0$. Then we take θ so that $\tan\theta=\alpha$.

4. Consider the 3^n n-tuples where each of the n coordinates is filled with a letter L,R or N. Each n-tuple with at least one L and at least one R corresponds to an ordered pair (A,B), where A is consisted of those j such that the j-th coordinate is L and B is consisted of those k such that the k-th coordinate is R.

   Then there are 2^n-1 n-tuples having only N or R in the coordinates with at least one R. Similarly, there are 2^n-1 n-tuples having only N or L in the coordinates with at least one L. The last n-tuple has N in all coordinates. These n-tuples do not form any $(A,B)\in Y_n$. So the number of elements in Y_n is $3^n-2(2^n-1)-1=3^n-2^{n+1}+1$.

5. From the power series of e^w, we have $f(z) = e^{z(z-1)^2}$. Observe that $f'(z) = e^{z(z-1)^2}(3z^2 - 4z + 1) = f(z)g(z)$, where $g(z) = 3z^2 - 4z + 1$. It follows that $f'' = f'g + fg'$, $f''' = f''g + 2f'g' + fg''$, $f^{(iv)} = f'''g + 3f''g' + 3f'g''$ (due to $g''' = 0$). By induction, we have

$$f^{(n+1)} = f^{(n)}g + a_n f^{(n-1)}g' + b_n f^{(n-2)}g'' \qquad (*)$$

for some integers a_n and b_n.

Suppose three consecutive coefficients are zero. Since the k-th coefficient is $f^{(k)}(0)/k!$ by Taylor's theorem, if $f^{(n)}(0) = f^{(n-1)}(0) = f^{(n-2)}(0) = 0$, then $f^{(n+1)}(0) = 0$ by (*) above. Hence, $f^{(k)}(0) = 0$ for all $k > n$. That implies $f(z)$ is a polynomial, which is false.

6. Let $z = 2x+iy$. We have $\dfrac{dz}{z} = \dfrac{d(2x+iy)}{2x+iy} = \dfrac{2x - iy}{4x^2+y^2}(2dx + idy) = \dfrac{4x\,dx + y\,dy}{4x^2+y^2} + i\left(\dfrac{2x\,dy - 2y\,dx}{4x^2+y^2}\right)$. Therefore, by the residue theorem, $\displaystyle\oint_C \frac{y\,dx - x\,dy}{4x^2+y^2} = -\frac{1}{2}\mathrm{Im}\left(\oint_C \frac{1}{z}\,dz\right) = -\frac{1}{2}(4\pi) = -2\pi$.