Fifth HKUST Undergraduate Math Competition – Junior Level

April 29, 2017

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1. Find all positive integers n for which $3n-4, 4n-5$ and $5n-3$ are all prime numbers.

Problem 2. Let $f:[0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)\ dx = \int_0^1 xf(x)\ dx = 1.$$

Prove that $\displaystyle\int_0^1 f^2(x)\ dx \geq 4$.

Problem 3. Determine all real numbers a such that the series $\displaystyle\sum_{n=1}^{\infty}\left(\cos\frac{a}{n}\right)^{n^3}$ is convergent.

Problem 4. Let $x_0, x_1, \ldots, x_n$ be distinct real numbers. Prove that there exist unique real numbers $a_0, a_1, \ldots, a_n$ such that for all polynomials $P(x)$ of degree n or less with real coefficients, we have

$$\int_0^1 P(t)\ dt = \sum_{j=0}^{n} a_j P(x_j).$$

Problem 5. Three husband-and-wife couples would like to sit around a table with six seats for dinner. If each of these people is randomly assigned one of the six seats, determine the probability that none of the couples would sit next to each other.

Problem 6. Let A and B be $n \times n$ real matrices satisfying $A^2 + B^2 = AB$. Prove that if $BA - AB$ is invertible, then n is divisible by 3.

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