Solution of the Fifth HKUST Undergraduate Math Competition – Junior Level

1. The sum of the three numbers is an even number. So at least one of them is even. The only even prime number is 2. Only $3n-4$ and $5n-3$ can be even. Solving the equations $3n-4=2$ and $5n-3=2$ yields $n=2$ and $n=1$ respectively. It is easy to check that $n=2$ makes all three given numbers prime.

2. Solution 1. From the Cauchy-Schwarz inequality $\Big(\int_0^1 f^2(x)\,dx\Big)\Big(\int_0^1 g^2(x)\,dx\Big) \geq \Big(\int_0^1 f(x)g(x)\,dx\Big)^2$, we set $g(x)=3x-1$ to get

$$\int_0^1 f^2(x)\,dx \geq \Big(\int_0^1 f(x)(3x-1)\,dx\Big)^2 \Big/ \Big(\int_0^1 (3x-1)^2\,dx\Big) = \frac{(3-1)^2}{\frac{9}{3}-\frac{6}{2}+1} = 4.$$

Solution 2. To relate $f(x), xf(x)$ and $f^2(x)$, consider $(f(x)+ax+b)^2 \geq 0$. Expanding, we get $f^2(x) \geq -a^2x^2 - b^2 - 2axf(x) - 2bf(x) - 2abx$. Then for all $a,b\in\mathbb{R}$,

$$\int_0^1 f^2(x)\,dx \geq \int_0^1 (-a^2x^2 - b^2 - 2axf(x) - 2bf(x) - 2abx)\,dx = -\frac{a^2}{3} - b^2 - 2a - 2b - ab.$$

Next, we consider maximizing $E(a,b) = -\dfrac{a^2}{3} - b^2 - 2a - 2b - ab$. For this purpose we first fix the real parameter b and consider the mapping $\phi(a) = -a^2/3 - (b+2)a - b^2 - 2b$, which achieves its maximum for $a_M = -3(b+2)/2$. Then $\phi(a_M) = 4-(b-2)^2/4$, which is maximum for $b=2$. The above argument implies that the maximum of $E(a,b)$ is 4 and it is attained for $a=-6$ and $b=2$. So the minimum is achieved for $f(x)=6x-2$.

Remark A faster way to finish is to solve $\partial E/\partial a = -2a/3-2-b=0$ and $\partial E/\partial b = -2b-2-a=0$ to get the critical point of E is at $(a,b)=(-6,2)$. Then $\displaystyle\int_0^1 f^2(x)\,dx \geq E(-6,2)=4$.

3. In the case $a=0$, the series diverges. For $a\neq 0$, we do some sketch work: $\cos\theta \approx 1-\theta^2/2$ as $\theta\to 0$ and $(1+x)^{1/x}\to e$ as $x\to 0$. So $\cos(a/n)\approx 1-a^2/(2n^2)$ and $(1-a^2/(2n^2))^{-2n^2/a^2}\to e$, hence $\big(\cos(a/n)\big)^{n^2} \approx (1-a^2/(2n^2))^{n^2} \to e^{-a^2/2} < 1$. Now more precisely,

$$\lim_{n\to\infty}\Big(\cos\frac{a}{n}\Big)^{n^2} = \lim_{n\to\infty}\exp\Big(n^2\ln\cos\frac{a}{n}\Big) = \exp\Big(\lim_{n\to\infty} n^2\ln\cos\frac{a}{n}\Big) = \exp\Big(\lim_{t\to 0}\frac{\ln\cos at}{t^2}\Big) = e^{-a^2/2} < 1.$$

So, there exists $c\in(0,1)$ such that for large n, $\Big(\cos\dfrac{a}{n}\Big)^{n^2} < c < 1$. Since $\displaystyle\sum_{n=1}^{\infty} c^n$ converges, so $\displaystyle\sum_{n=1}^{\infty}\Big(\cos\frac{a}{n}\Big)^{n^3}$ converges for all nonzero real numbers $a\neq 0$.

4. Solution 1. For polynomial $P(t)=c_0+c_1t+\cdots+c_nt^n$, we have

$$\int_0^1 P(t)\,dt = \sum_{i=0}^{n}\frac{c_i}{i+1} = \sum_{i=0}^{n} c_i\Big(\sum_{j=0}^{n} a_j x_i^j\Big) = \sum_{j=0}^{n} a_j\Big(\sum_{i=0}^{n} c_i x_j^i\Big) = \sum_{j=0}^{n} a_j P(x_j)$$

if we can find $a_0, a_1, \ldots, a_n$ such that $\begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ \vdots \\ 1/(n+1) \end{pmatrix}.$

Now the $(n+1)\times(n+1)$ Vandermonde matrix on the left is knowm to have nonzero determinant $\displaystyle\prod_{1\leq i<j\leq n}(x_j-x_i)$. Therefore, the matrix is invertible and the a_i's exist.

Solution 2. The set V_{n+1} of all real polynomials of degree n or less is a real vector space of dimension $n+1$. The map $T : V_{n+1} \to \mathbb{R}^{n+1}$ defined by $T(P) = (P(x_0), P(x_1), \ldots, P(x_n))$ is a linear transformation with null space equals $\{0\}$ due to degree of $P(x)$ is at most n. Then T is invertible.

Define $f : \mathbb{R}^{n+1} \to \mathbb{R}$ by $f(w) = \int_0^1 (T^{-1}(w))(t)\ dt$ for all $w = (w_0, w_1, \ldots, w_n)$ in $\mathbb{R}^{n+1}$. Then f is a linear functional. Hence, there is a unique vector $a = (a_0, a_1, \ldots, a_n)$ in $\mathbb{R}^{n+1}$ such that $f(w) = a_0w_0 + a_1w_1 + \cdots + a_nw_n$ for all $w = (w_0, w_1, \ldots, w_n)$. Letting $P = T^{-1}(w)$, we have $(P(x_0), P(x_1), \ldots, P(x_n)) = T(P) = w = (w_0, w_1, \ldots, w_n)$ and

$$\int_0^1 P(t)\ dt = \int_0^1 (T^{-1}(w))(t)\ dt = f(w) = \sum_{j=0}^{n} a_j w_j = \sum_{j=0}^{n} a_j P(x_j).$$

5. We number the couples as 1,2,3. Let I be the set of arrangements of seats for the six people. Then $|I| = 6!$. Let A_i be the set of arrangements such that the i-th couple sit next to each other. It is easy to see that

$$|A_i| = (6 \cdot 2) \cdot 4!, \quad |A_i \cap A_j| = (6 \cdot 2) \cdot (3 \cdot 2) \cdot 2 \ \text{ for } 1 \le i < j \le 3$$

and $|A_1 \cap A_2 \cap A_3| = 6 \cdot 2^4$.

Define $A^c = I \setminus A$ for any $A \subseteq I$. Then the total number of arrangements such that none of the couples sit next to each other is

$$\begin{aligned} |A_1^c \cap A_2^c \cap A_3^c| &= |I| - |A_1 \cup A_2 \cup A_3| \\ &= |I| - \sum_{i=1}^{3} |A_i| + \sum_{1 \le i < j \le 3} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3| \\ &= 6! - 3 \cdot (6 \cdot 2) \cdot 4! + 3 \cdot (6 \cdot 2) \cdot (3 \cdot 2) \cdot 2 - 6 \cdot 2^4 = 192. \end{aligned}$$

Then the desired probability is $|A_1^c \cap A_2^c \cap A_3^c|/6! = 4/15$.

6. (*Remark* To start, a possible approach is to consider $a^2 + b^2 = ab$ as $a^2 - ab + b^2 = (a^3 + b^3)/(a + b) = (a + \omega b)(a + \omega^2 b)$, where ω is a cube root of unity.)

Let $\omega = (-1 + i\sqrt{3})/2$, then $\omega^3 = 1$ and $\omega^2 = \overline{\omega}$. Let $S = A + \omega B$ and $T = A + \overline{\omega}B$. We have

$$ST = (A + \omega B)(A + \overline{\omega}B) = A^2 + \omega BA + \overline{\omega}AB + B^2 = AB + \omega BA + \overline{\omega}AB = \omega(BA - AB),$$

where we used $\overline{\omega} + 1 = -\omega$. Since $\det T = \overline{\det S}$, we have $\det(ST) = \det S \det T = |\det S|^2$. Since $\det(BA - AB) \in \mathbb{R} \setminus \{0\}$,

$$\det(ST) = \det(\omega(BA - AB)) = \omega^n \det(BA - AB).$$

So ω^n is real. Therefore, n is divisible by 3.