Fifth HKUST Undergraduate Math Competition – Senior Level

April 29, 2017

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1. Suppose $a_1, a_2, a_3, \ldots$ is an infinite sequence of positive integers. Define a new sequence $b_1, b_2, b_3, \ldots$ by:

$$b_1 = a_1,\ b_2 = a_2 b_1 + 1 \text{ and for } k \geq 3,\ b_k = a_k b_{k-1} + b_{k-2}.$$

Prove that no two consecutive b_k's are even.

Problem 2. Let n be an integer greater than 2. A deck of n cards with 3 aces and $n-3$ kings is shuffled, with all permutations equally probable. The cards are then turned over one after the other until two aces have appeared with the second ace being the k-th card. Show that the expected value of k is $(n+1)/2$.

Problem 3. Determine with proof whether or not the sequence

$$\sum_{n=1}^{\infty} \sin(\pi\sqrt{n^2+1})$$

converges.

Problem 4. Let M be an $n \times n$ matrix over the real numbers R. Prove that

$$\operatorname{rank} M^2 \leq \frac{\operatorname{rank} M + \operatorname{rank} M^3}{2}.$$

Problem 5. Suppose $p(z), q(z)$ and $r(z)$ are continuous functions defined on $\mathbb{C}$ such that whenever $|z| = 1$, we have $p(z), q(z), r(z) \in \mathbb{R}$ and

$$4e^{p(z)+r(z)} \leq q(z)^2.$$

Show that there does not exist any entire function $f(z)$ such that

$$z^2 f(z)^2 e^{p(z)} + z f(z) q(z) + e^{r(z)} = 0 \quad \text{on } \{z \in \mathbb{C} : |z| = 1\}.$$

Problem 6. Let p be a prime number. For a group G of order p^4, suppose the center of G has order p^2. A *conjugacy class* of G is a set $\{g^{-1}xg : g \in G\}$ for some $x \in G$. Determine the number of distinct conjugacy classes in G.

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