Solution of the Fifth HKUST Undergraduate Math Competition – Senior Level

1. Let S_k be the statement that b_k and b_{k+1} are not both even. Note that if $b_1 = a_1$ is even, then $b_2 = a_2b_1 + 1$ is odd, so the statement S_1 is true. Suppose some S_k is false. Let k_0 be the smallest index for which S_{k_0} is false, i.e., both b_{k_0} and b_{k_0+1} are even. Since S_1 is true, $k_0 > 1$. Then S_{k_0-1} is true, i.e., the pair b_{k_0-1}, b_{k_0} is not both even. Since b_{k_0} is even, we must have b_{k_0-1} is odd. Buth then $b_{k_0+1} = a_{k_0+1}b_{k_0} + b_{k_0-1}$ is odd, which contradicts the hypothesis S_{k_0} is false. Therefore there is no index k for which statement S_k is false.

2. <u>*Solution 1.*</u> For each permutations σ of the n cards, pair σ with its reverse permutation σ^*. Let the second ace be at the k_σ-th card in the σ case. Then the second ace will be at the k_{σ^*}-th card in the σ^* case, where $k_{\sigma^*} = (n - k_\sigma + 1)$. So the expected value of k is $(k_\sigma + k_{\sigma^*})/2 = (n+1)/2$.

   <u>*Solution 2.*</u> Let $X : \Omega \to \mathbb{N}$ be the random variable with $X = k$ if the second ace comes at the k-th card, whence $p(X = k) = (k-1)(n-k)/\binom{n}{3}$ with $\sum_{k=2}^{n-1}(k-1)(n-k) = \binom{n}{3}$. Let $S = \binom{n}{3}EX = \sum_{k=2}^{n-1} k(k-1)(n-k)$. Replacing the running index k with $n+1-k$, we obtain $2S = (n+1)\sum_{k=2}^{n-1}(k-1)(n-k)$. Then $EX = (n+1)/2$.

3. For $x \geq 0$, define a function f by
$$f(x) = \begin{cases} \frac{\sqrt{1+x^2}-1}{x} & x > 0 \\ 0 & x = 0 \end{cases}.$$
   One can easily check that $\lim\limits_{x\to 0^+} f(x) = 0$. So the function is continuous on $[0, +\infty)$. Furthermore, for $x > 0$, we have
$$f'(x) = \frac{1}{x^2}\Big(1 - \frac{1}{\sqrt{1+x^2}}\Big) > 0.$$
   This means f is an increasing function for $x \geq 0$. (In fact, $\lim\limits_{x\to 0^+} f'(x) = \dfrac{1}{2}$ and the right-hand derivative at 0 is $\frac{1}{2}$.) Now,
$$\begin{aligned}
\sin(\pi\sqrt{n^2+1}) &= \sin\Big(\pi n\sqrt{1+1/n^2}\Big) \\
&= \sin\Big(\pi n\big(1 + \sqrt{1+1/n^2} - 1\big)\Big) = \sin\Big(\pi n + \pi\frac{\sqrt{1+1/n^2}-1}{1/n}\Big) \\
&= (-1)^n \sin\Big(\pi\frac{\sqrt{1+1/n^2}-1}{1/n}\Big) = (-1)^n \sin\Big(\pi f\big(\frac{1}{n}\big)\Big).
\end{aligned}$$
   The sequence $\sin\big(\pi f(\frac{1}{n})\big)$ is decreasing with limit zero. By the alternating series test, the sequence converges.

4. Let $V = \mathbb{R}^n$. Then M^2V is a vector subspace of MV. The dimension of the quotient space MV/M^2V is $\operatorname{rank} M - \operatorname{rank} M^2$. Similarly, the dimension of the quotient space M^2V/M^3V is $\operatorname{rank} M^2 - \operatorname{rank} M^3$.

   Now M induces a linear transformation from MV/M^2V onto M^2V/M^3V due to $M(Mx+M^2V) = M^2x + M^3V$. So $\dim(M^2V/M^3V) \leq \dim(MV/M^2V)$, which yields $\operatorname{rank} M^2 - \operatorname{rank} M^3 \leq \operatorname{rank} M - \operatorname{rank} M^2$. Then $\operatorname{rank} M^2 \leq (\operatorname{rank} M + \operatorname{rank} M^3)/2$.

5. Assume such f exists. Then $zf(z)$ satisfies the quadratic equation (in the variable w)
$$e^{p(z)}w^2 + q(z)w + e^{r(z)} = 0.$$
   When $|z| = 1$, the given equations on p, q and r guarantee that the roots w of the above quadratic equation must be real. Therefore, on $C = \{z : |z| = 1\}$, the value $zf(z)$ must be real.

Note that $zf(z)$ is also an entire function. Hence, the real-valued function $v(x,y) = \mathrm{Im}(zf(z))$, where $z = x+iy$, is harmonic by the Cauchy-Riemann equations.

Moreover, $zf(z)$ is real valued on C. So the harmonic function $v(x,y) = 0$ on C. By the maximum principle, $v(x,y) = 0$ for $x^2 + y^2 < 1$. Stardard argument using the Cauchy-Riemann equations then shows $zf(z)$ is a real constant function for $x^2 + y^2 < 1$.

Now the constant must be 0 since $zf(z) = 0$ when $z = 0$. Hence, $f(z) = 0$ on the annulus $\{z : 0 < |z| < 1\}$. By the identity theorem, $f \equiv 0$ on $\mathbb{C}$. Then $e^{r(z)} = 0$ on $\mathbb{C}$, which is a contradiction.

6. For x in the center of G, the conjugacy class of x is just $\{x\}$. For x not in the center of G, the normalizer $N_x(G) = \{g \in G : xg = gx\}$ of x contains x and the center of G. Now $N_x(G)$ is not equal to G due to x not in the center. So $N_x(G)$ has at least p^2+1 elements and less than p^4 elements. Then the order of $N_x(G)$ can only be p^3. The number of elements in the conjugacy class of x is the index of $N_x(G)$, which is $p^4/p^3 = p$. Therefore, the number of conjugacy classes of G is $p^2 + (p^4 - p^2)/p = p^2 + p^3 - p$.