Sixth HKUST Undergraduate Math Competition – Junior Level

April 28, 2018

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1. Find $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)(1+x^8)}$. Show details.

Problem 2. Let A be a real orthogonal $n\times n$ matrix. Determine for which positive integers n there exists a real orthogonal $n\times n$ matrix B such that $A+B$ is a real orthogonal matrix.

Problem 3. Let $f:[0,1]\to(0,+\infty)$ be continuous and strictly decreasing. Prove that

$$\frac{\int_0^1 xf^2(x)\ dx}{\int_0^1 xf(x)\ dx} \le \frac{\int_0^1 f^2(x)\ dx}{\int_0^1 f(x)\ dx}.$$

Problem 4. In $\mathbb{R}^2$, let D be a closed disk with positive radius and center at $(0,0)$. Prove that for every (a,b) in $\mathbb{R}^2$, there exists a positive integer n such that the set $S=\{(x+na,y+nb):(x,y)\in D\}$ contains an element (p,q), where p and q are integers.

Problem 5. Let $f:(-1,1)\to\mathbb{R}$ be a function of the form $f(x)=\displaystyle\sum_{i=0}^\infty c_ix^i$, where each coefficient $c_i\in\{0,1,2\}$. If $f\Big(\dfrac{4}{5}\Big)=\dfrac{5}{4}$, then prove that $f\Big(\dfrac{1}{3}\Big)$ is an irrational number.

Problem 6. Let $k_1,k_2,k_3,\ldots$ be a sequence of strictly increasing positive integers such that $\displaystyle\lim_{n\to\infty}\frac{k_n}{n}=+\infty$. Prove that $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{[\sqrt{n}]}}{k_n}$ converges, where $[x]$ is the greatest integer less than or equal to x.

– End of Paper –