Solution of the Sixth HKUST Undergraduate Math Competition – Junior Level

1. For $t > 0$, define $I(t) = \displaystyle\int_{1/t}^{t} \frac{dx}{(1+x^2)(1+x^8)}$. Substituting $x = \dfrac{1}{u}$, we have $I(t) = \displaystyle\int_{1/t}^{t} \frac{u^8\ du}{(u^2+1)(u^8+1)}$. Adding these, we get $2I(t) = \displaystyle\int_{1/t}^{t} \frac{du}{1+u^2}$. Dividing by 2 and letting t go to ∞, we get the answer is $\dfrac{1}{2}\displaystyle\int_0^\infty \frac{du}{1+u^2} = \frac{1}{2}\operatorname{Arctan} u\Big|_0^\infty = \frac{\pi}{4}$.

2. Such a B matrix exists if and only if there exists a real orthogonal $n \times n$ matrix V so $A + AV$ is orthogonal, i.e. $A(I+V)(I+V^T)A^T = I$, which is the same as $V^2 + V + I = 0$. No matrix with a real eigenvalue satisfies this, hence no solution if n is odd. For n even, let $M = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$ and $V = \operatorname{diag}(M, M, \ldots, M)$ with M occuring $n/2$ times will work as $V^2 + V + I = 0$ by a checking.

3. The conclusion is equivalent to $I = \displaystyle\int_0^1 f^2(x)\ dx \int_0^1 yf(y)\ dy - \int_0^1 f(x)\ dx \int_0^1 yf^2(y)\ dy \geq 0$. Now
$$I = \int_0^1\int_0^1 f(x)f(y)y\Big(f(x)-f(y)\Big)\ dx\ dy \quad \text{and} \quad I = \int_0^1\int_0^1 f(y)f(x)x\Big(f(y)-f(x)\Big)\ dx\ dy.$$
Adding these, we get $2I = \displaystyle\int_0^1\int_0^1 f(x)f(y)(y-x)\Big(f(x)-f(y)\Big)\ dx\ dy$. Since $f > 0$ and f is strictly decreasing on $[0,1]$, for all $x, y \in [0,1]$, we have $(y-x)(f(x)-f(y)) \geq 0$. So $2I \geq 0$. Therefore, $I \geq 0$.

4. Let r be the radius of D and s be an integer greater than $\sqrt{2}/r$. Consider the points $A_k = (ka - [ka], kb - [kb])$ for $k = 0, 1, 2, \ldots, s^2$. Since these k^2 points are in $W = [0,1] \times [0,1]$, there exist two of them, say A_i, A_j with $j > i$, in a square with side $1/s$. Then the segment A_iA_j has length at most $\sqrt{2}/s$, which is less than r. Let $n = j - i$, $p = [ja] - [ia]$ and $q = [jb] - [ib]$. Then p, q are integers and (na, nb) and (p, q) has distance
$$\sqrt{(na-p)^2 + (nb-q)^2} = \sqrt{(ja - [ja] - ia + [ia])^2 + (jb - [jb] - ib + [ib])^2} \leq \sqrt{2}/s < r.$$

5. Assume $f\Big(\dfrac{1}{3}\Big)$ is rational. Then $f\Big(\dfrac{1}{3}\Big) = \displaystyle\sum_{i=0}^{\infty} c_i 3^{-i}$ is the base 3 expansion of a rational number. It must be eventually periodic. Hence, there exist positive integer m, n with $c_i = c_{m+i}$ for all $i \geq n$. Then
$$f(x) = \sum_{i=0}^{n-1} c_i x_i + \frac{x^n}{1-x^m}\sum_{i=0}^{m-1} c_{n+i}x^i.$$
At $x = \dfrac{4}{5}$, we have $\dfrac{5}{4} = f\Big(\dfrac{4}{5}\Big) = \dfrac{1}{5^n}\displaystyle\sum_{i=0}^{n-1} c_i 4^i 5^{n-i} + \frac{4^n 5^m}{5^{n+m}(5^n - 4^m)}\sum_{i=0}^{m-1} c_{n+i}4^i 5^{n+m-i}$. All terms on the right-hand side have odd denominators. Then, the denominator of the sum must be odd, which is a contradiction.

6. From $\displaystyle\lim_{n\to\infty}\Big(\frac{1}{k_n\sqrt{n}}\Big/\frac{1}{n\sqrt{n}}\Big) = \lim_{n\to\infty}\frac{n}{k_n} = 0$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$ converges, we get $\displaystyle\sum_{n=1}^{\infty}\frac{1}{k_n\sqrt{n}}$ converges by the limit comparison test. Let $a_n = \dfrac{1}{k_n}$. We claim $S_n = \displaystyle\sum_{i=1}^{n}(-1)^{[\sqrt{i}]}a_i$ is a Cauchy sequence. To bound

$S_n - S_{m-1} = \sum_{i=m}^{n} (-1)^{[\sqrt{i}]} a_i$ from above, we delete all terms with $[\sqrt{i}] = [\sqrt{m}]$ if $[\sqrt{m}]$ is odd and insert all missing terms with $[\sqrt{i}] = [\sqrt{m}]$ if $[\sqrt{m}]$ is even. Similarly delete or insert terms with $[\sqrt{i}] = [\sqrt{n}]$. (If $[\sqrt{n}] = [\sqrt{m}]$ and this is odd, then this tells us to delete all terms to get 0 as an upper bound; otherwise assume some terms remain.) Thus, for a fixed n, taking $1 < m < n$, there exist i, j with $i \le j$ such that

$$\begin{aligned}
S_n - S_{m-1} = \sum_{t=m}^{n} (-1)^{[\sqrt{t}]} a_t \ &\le \sum_{t=(2i)^2}^{(2j+1)^2-1} (-1)^{[\sqrt{t}]} a_t \\
&\le \sum_{t=(2i)^2}^{(2i+1)^2-1} a_t + \sum_{t=i+1}^{j} \Big[2a_{(2t+1)^2-2} + \sum_{u=0}^{(2t)^2-(2t-1)^2-1} \left(a_{(2t)^2+u} - a_{(2t-1)^2+u} \right) \Big] \\
&\le (4i+1)a_{(2i)^2} + \sum_{t=i+1}^{j} 2a_{(2t+1)^2-2} \le (4i+1)a_{(2i)^2} + 2 \sum_{t=i+1}^{j} \frac{1}{8t} \sum_{u=(2t-1)^2-1}^{(2t+1)^2-2} a_u \\
&\le (4i+1)a_{(2i)^2} + \sum_{t=i+1}^{j} \sum_{u=(2t-1)^2-1}^{(2t+1)^2-2} \frac{a_u}{\sqrt{u}} \le (4i+1)a_{(2i)^2} + \sum_{u=(2i+1)^2-1}^{(2j+1)^2-2} \frac{a_u}{\sqrt{u}}.
\end{aligned}$$

Note that $i \to \infty$ as $m \to \infty$. We claim the last expression goes to 0 as $i \to \infty$. The second term goes to 0 as $i \to 0$ due to $\sum a_u/\sqrt{u}$ converges. So we just need to show the first term also goes to 0. Suppose not. For some $\varepsilon > 0$, assume $(4i+1)a_{(2i)^2} > \varepsilon$ infinitely often. Recursively take $i_t > 2i_{t-1}$ for $t \ge 1$, with $(4i_t+1)a_{(2i_t)^2} > \varepsilon$. Now

$$\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} \ge \sum_{t=1}^{\infty} \sum_{u=i_t^2}^{(2i_t)^2} \frac{a_u}{\sqrt{u}} \ge \sum_{t=1}^{\infty} 3i_t^2 \frac{a_{(2i_t)^2}}{2i_t} = \frac{3}{2} \sum_{t=1}^{\infty} i_t a_{(2i_t)^2} \ge \frac{3}{10} \sum_{t=1}^{\infty} (4i_t+1) a_{(2i_t)^2} \ge \frac{3}{10} \sum_{u=1}^{\infty} \varepsilon = +\infty,$$

which is a contradiction. So $\lim_{n\to\infty} \sup\{S_n - S_{m-1} : m < n\} \le 0$. Similarly, we can bound $S_n - S_{m-1} = \sum_{t=m}^{n} (-1)^{[\sqrt{t}]} a_t \ge \sum_{t=(2i-1)^2}^{(2j)^2-1} (-1)^{[\sqrt{t}]} a_t$ from below to conclude that $\lim_{n\to\infty} \inf\{S_n - S_{m-1} : m < n\} \ge 0$. So S_n is a Cauchy sequence, hence converges.