Sixth HKUST Undergraduate Math Competition – Senior Level

April 28, 2018

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1. Let n be a positive integer and A, B be $n \times n$ matrices over the complex numbers. Prove that A and B have a common eigenvalue if and only if $AX = XB$ for some $n \times n$ matrix $X \neq 0$.

Problem 2. Find all continuous functions $y : [0, \infty) \to \mathbb{R}$ such that $y(0) = 0$, y is differentiable on $(0, \infty)$ satisfying $y'(x) = \int_0^x \sin(y(u))\, du + \cos x$ for all $x > 0$.

Problem 3. Let a and b be positive integers with $a > 1$. If a and b are both odd or both even, then prove that $2^a - 1$ does not divide $3^b - 1$.

Problem 4. Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous such that for every $x \in [0, 1]$ and $y_0 \neq y_1$ in $\mathbb{R}$, we have

$$\frac{1}{2} \leq \frac{f(x, y_0) - f(x, y_1)}{y_0 - y_1} \leq \frac{3}{2}.$$

Prove that there exists a unique real-valued continuous function h on $[0, 1]$ such that for all $x \in [0, 1]$, $f(x, h(x)) = 0$.

Problem 5. Let $u \cdot v$ denote the usual inner product of $u, v \in \mathbb{R}^n$. For positive integer $k < n$, let $G(k, n)$ be the set of all k-dimensional linear subspaces in $\mathbb{R}^n$. For $v \in \mathbb{R}^n$ and a linear subspace S in $\mathbb{R}^n$, let $d(v, S)$ denote the usual distance from v to S. For $V \in G(k, n)$, let $B(V) = \{v \mid v \in V, v \cdot v = 1\}$. For $V, U \in G(k, n)$, let $d(V, U) = \max\{d(v, U) \mid v \in B(V)\}$.

(a) Prove that for $V, W, U \in G(k, n)$, $d(V, U) \leq d(V, W) + d(W, U)$.

(b) Let $\{v_1, v_2, \ldots, v_k\}$, $\{w_1, w_2, \ldots, w_k\}$ be orthonormal basis of $V, W \in G(k, n)$ respectively. Let A be the $k \times k$ matrix with (i, j) entry equal $v_i \cdot w_j$. Let λ be the smallest eigenvalue of AA^T. Determine the value of $d(V, W)$ in terms of λ.

(c) Prove that $d(V, W) = d(W, V)$ for all $V, W \in G(k, n)$.

Problem 6. Let $S = \{z \mid z \in \mathbb{C},\ 0 < |z| < 2\}$ and $f : S \to \mathbb{C}$ be a holomorphic function such that $\operatorname{Re} f(z) \geq 0$ and $\operatorname{Im} f(z) \geq 0$ for all $z \in S$. Prove that f has a removable singularity at 0.

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