Solution of the Sixth HKUST Undergraduate Math Competition – Senior Level

1. Suppose c is a common eigenvalue of A and B. Since B and B^T have the same eigenvalues due to similarity, we have $Av = cv$ and $B^T w = cw$ for some nonzero column vectors v and w. Let $X = vw^T$. Then $X \neq 0$ and $AX = cvw^T = vw^T B = XB$.

   Conversely, suppose $AX = XB$ for some $X \neq 0$. By induction, $f(A)X = Xf(B)$ for every polynomial f. Take $f(z) = \det(zI - B) = (z - z_1)(z - z_2)\cdots(z - z_n)$. By the Cayley-Hamilton's Theorem, $f(B) = 0$. So $(A - z_1 I)(A - z_2 I)\cdots(A - z_n I)x = 0$ for some $x \neq 0$. This implies $A - z_r I$ is singular for some r. Then z_r is an eigenvalue of A and B.

2. The function $y(x) = x$ is a solution. Suppose $w(x)$ is another solution. Then

$$|y'(x) - w'(x)| \leq \int_0^x |\sin y(u) - \sin w(u)|\ du \leq \int_0^x |y(u) - w(u)|\ du.$$

   If $t = \inf\{x > 0 : y(x) \neq w(x)\} < \infty$, then $0 < c = \sup\{|y(x) - w(x)| : x \leq t + 1\} < \infty$. However, for $x \leq t$, $y(x) = w(x)$ and for $t \leq x \leq t + 1$,

$$\begin{aligned} |y(x) - w(x)| &= \Big|\int_t^x (y'(s) - w'(s)\ ds\Big| \leq \int_t^x \int_0^s |y(u) - w(u)|\ du\ ds \\ &= \int_t^x \int_t^s |y(u) - w(u)|\ du\ ds \leq \frac{c(x-t)^2}{2} \leq \frac{c}{2} \end{aligned}.$$

   Thus, $c \leq c/2$, a contradiction. So $y(x) = x$ is the unique solution.

   Remarks. For those who used a uniqueness theorem, they need to provide and check the relevant condition, such as Lipschitz' condition.

3. If a is even, then 3 divides $2^a - 1$, but 3 does not divide $3^b - 1$. So $2^a - 1$ does not $3^b - 1$. If $a > 1$ is odd, then $m = 2^a - 1 \equiv 1 \pmod 3$. Hence, $\left(\dfrac{m}{3}\right) = +1$ in the Legendre symbol (as well as the Jacobi symbol). By quadratic reciprocity, $\left(\dfrac{3}{m}\right) = \left(\dfrac{m}{3}\right)(-1)^{(m-1)(3-1)/4}$. Now $(m-1)/2 = 2^{a-1} - 1$ is odd and so is $(3-1)/2$. Then $\left(\dfrac{3}{m}\right) = -\left(\dfrac{m}{3}\right) = -1$. Thus, 3 is a quadratic nonresidue $(\text{mod } m)$.

   Assume $2^a - 1 \mid 3^b - 1$ with $b = 2n - 1$. It follows that $m \mid 3^{2n} - 3$. Therefore, $(3^n)^2 \equiv 3 \pmod m$. However, this implies that 3 is a quadratic residue $(\text{mod } m)$, which is a contradiction. Therefore, $2^a - 1$ cannot divide $3^b - 1$ if a and b are odd.

4. Define $T : C[0,1] \to C[0,1]$ by $(Tg)(x) = g(x) - f(x, g(x))$. For every $x \in [0,1]$, if $g(x) = h(x)$, then $|(Tg)(x) - (Th)(x)| = 0$. Otherwise,

$$\begin{aligned} |(Tg)(x) - (Th)(x)| &= |g(x) - f(x, g(x)) - h(x) + f(x, h(x))| \\ &= \Big|1 - \frac{f(x, g(x)) - f(x, h(x))}{g(x) - h(x)}\Big||g(x) - h(x)| \\ &\leq \frac{1}{2}|g(x) - h(x)|. \end{aligned}$$

   So $\|Tg - Th\| \leq \frac{1}{2}\|g - h\|$. By the contractive mapping theorem, T has a unique fixed point h. Then $T(h) = h$ and $f(x, h(x)) = 0$ for all $x \in [0,1]$.

5. (a) By compactness of $B(V)$ and continuity of d, there exists $v_0 \in B(V)$ satisfying $d(v_0, U) = d(V, U)$. Let $w \in W$ be the orthogonal projection of v_0 onto W. Then $d(v_0, w) = d(v_0, W) \leq d(V, W)$. Since $w \cdot w \leq 1$, $d(w, U) \leq d(w/|w|, U) \leq d(W, U)$. So

$$d(V, U) = d(v_0, U) \leq d(v_0, w) + d(w, U) \leq d(V, W) + d(W, U).$$

(b) Let $x = x_1v_1 + x_2v_2 + \cdots + x_kv_k \in B(V)$. The projection of x onto W is

$$(x \cdot w_1)w_1 + (x \cdot w_2)w_2 + \cdots + (x \cdot w_k) = \sum_{i,j=1}^{k} (v_i \cdot w_j)x_iw_j.$$

Then $d(x, W)^2 = x \cdot x - \Big(\sum_{i,j=1}^{k} (v_i, w_j)x_iw_j\Big) \cdot \Big(\sum_{i,j=1}^{k} (v_i, w_j)x_iw_j\Big) = 1 - [x_1 \ \cdots \ x_k]AA^T[x_1 \ \cdots \ x_k]^T.$

Now AA^T is symmetric, hence diagonalizable. So there is a unitary matrix P such that $AA^T = P^{-1}MP$, where M is the diagonal matrix with the eigenvalues of AA^T as entries in increasing order $\lambda = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$. For all $x = x_1v_1 + x_2v_2 + \cdots + x_nv_n \in B(V)$, $xMx^T = \lambda_1x_1^2 + \cdots + \lambda_kx_k^2 \geq \lambda_1(x_1^2 + \cdots + x_k^2) = \lambda$. So the minimum of xMx^T is λ and for unit eigenvectors $(x_1, x_2, \ldots, x_k)$ of λ, $d(V, W) = d(x, W) = \sqrt{1 - \lambda}$.

(c) From (b), we have $d(W, V) = \sqrt{1 - \alpha}$, where α is the least eigenvalue of A^TA. Since $(A^TA)^T = A^TA$, so $\alpha = \lambda$ and $d(V, W) = d(W, V)$.

6. Suppose f is not constant, then the range of f is an open subset of the first quadrant. Let g be a Möbius map from the open first quadrant to the open unit disk. We see $g \circ f$ is bounded. Then 0 is a removable singularity of $g \circ f$. So $g \circ f$ can be extended to a holomorphic map on $S \cup \{0\}$. Then $f = g^{-1} \circ (g \circ f)$ has a removable singularity at 0.

Alternatively, let the Laurent series of f on S be $\displaystyle\sum_{k=-\infty}^{\infty} a_kz^k$. We will show $a_{-n} = 0$ for all $n \geq 1$. Consider $I_\pm = \displaystyle\int_0^{2\pi} f(re^{i\theta})(1 \pm \cos n\theta)\, d\theta$. From the given condition, observe that $1 \pm \cos n\theta \geq 0$. We get $\operatorname{Re} I_\pm$ and $\operatorname{Im} I_\pm$ are both nonnegative for all $r \in (0, 2)$ and $n \geq 1$. Next we will express the above integral as a contour integral. For $r \in (0, 2)$, we consider the change of variable $z = re^{i\theta}$ with $\theta \in [0, 2\pi]$ and apply the residue theorem to get

$$I_\pm = \int_{|z|=r} f(z)\Big[1 \pm \frac{1}{2}\Big(\frac{z^n}{r^n} + \frac{r^n}{z^n}\Big)\Big]\frac{dz}{iz} = 2\pi\Big[a_0 \pm \frac{1}{2}\Big(\frac{a_{-n}}{r^n} + a_nr^n\Big)\Big].$$

Since $\operatorname{Re} I_\pm \geq 0$ and $\operatorname{Im} I_\pm \geq 0$, we get

$$\operatorname{Re}\Big[a_0r^n \pm \frac{1}{2}(a_{-n} + a_nr^{2n})\Big] \geq 0 \quad \text{and} \quad \operatorname{Im}\Big[a_0r^n \pm \frac{1}{2}(a_{-n} + a_nr^{2n})\Big] \geq 0$$

for all $r \in (0, 2)$ and $n \geq 1$. By letting $r \to 0^+$, we conclude that $\operatorname{Re} a_{-n} = \operatorname{Im} a_{-n} = 0$ for all $n \geq 1$. Hence, $a_{-n} = 0$ for all $n \geq 1$. Therefore, at 0, f has a removable singularity.