Seventh HKUST Undergraduate Math Competition – Junior Level

April 27, 2019

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1. For a real number w, $[w]$ denotes the greatest integer less than or equal to w. Let $P(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial of degree $n \geq 2$ such that

$$0 < a_0 < -\sum_{k=1}^{[n/2]} \frac{a_{2k}}{2k+1}.$$

Prove that $P(x)$ has a root in $(-1, 1)$.

Problem 2. For real numbers x satisfying $0 < |x| < 1$,

(1) prove that if $(1-x)^{1-\frac{1}{x}} < (1+x)^{\frac{1}{x}}$, then $(1-x^2)^{1+\frac{1}{x}} < 1-x < (1-x^2)^{\frac{1}{x}}$;

(2) prove that $(1-x)^{1-\frac{1}{x}} < (1+x)^{\frac{1}{x}}$.

Problem 3. In $\mathbb{R}^2$, let C_0 and C_1 be two circles of radius $1/2$ centered at $(0, 1/2)$ and $(1, 1/2)$ respectively. Let C_2 be the circle that is tangent to C_0, C_1 and the x-axis. For $n \geq 2$, let C_{n+1} be the circle different from C_{n-2} that is tangent to C_n, C_{n-1} and the x-axis. Let $(x_n, 0)$ be the point where C_n is tangent to the x-axis. Determine the limit of x_n as n tends to infinity.

Problem 4. Let $T = \{a_1, -a_1, a_2, -a_2, \ldots, a_n, -a_n\}$ be a set of $2n$ distinct integers. Let $1 \leq m < 2^n$. Prove that there exists a nonempty subset S of T such that for each $i = 1, 2, \ldots, n$, the integers a_i and $-a_i$ are not both in S and the sum of all elements of S is divisible by m.

Problem 5. Let $f : [0, 1] \to \mathbb{R}$ be continuous. If f is differentiable on $(0, 1)$, $f(0) = 0$ and $0 < f'(x) \leq 1$ for all $x \in (0, 1)$. Prove that $\left(\int_0^1 f(x)\, dx\right)^2 \geq \int_0^1 f^3(x)\, dx$.

Problem 6. Let n be a positive integer and

$$f(x) = \frac{x^2(2 \cdot 3 \cdot n - x)}{2^5 \cdot 3^3 \cdot n^2}.$$

Find the number of distinct integers among $[f(0)], [f(1)], [f(2)], \ldots, [f(36n)]$ in terms of n, where $[x]$ denotes the greatest integer less than or equal to x.

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