Solutions of 2019 UG Math Competition - Junior Level

Problem 1. Since $\int_{-1}^{1} P(x)\,dx = 2\sum_{k=0}^{[n/2]} \frac{a_{2k}}{2k+1} < 0$, there exists $x_0 \in (-1,1)$ such that $P(x_0) < 0$. Since $P(0) = a_0 > 0$, by the intermediate value theorem, $P(x)$ has a root in $(-1,1)$.

Problem 2. (1) For all $0 < |x| < 1$, if (*) $(1-x)^{1-1/x} < (1+x)^{1/x}$, then multiplying both sides of (*) by $(1+x)^{1-1/x}$, we get $(1-x^2)^{1-1/x} < 1+x$. Then, replacing x by $-x$, it becomes $(1-x^2)^{1+1/x} < 1-x$. Next, multiplying both sides of (*) by $(1-x)^{1/x}$, we also get $1-x < (1-x^2)^{1/x}$.

(2) Multiplying both sides of (*) by $(1-x)^{1/x}$, we get $1-x < (1-x^2)^{1/x}$. Then taking log on both sides, this is equivalent to $\log(1-x) < \frac{1}{x}\log(1-x^2)$. For $0 < x < 1$, we get $x\log(1-x) < \log(1-x^2)$. For $-1 < x < 0$, we get $x\log(1-x) > \log(1-x^2)$.

Let $f(x) = \log(1-x^2) - x\log(1-x)$. Then $f'(x) = \frac{1}{1+x} - 1 - \log(1-x) \geq 0$ for $x \in (-1,1)$ and $f''(x) = \frac{x(x+3)}{(1+x)^2(1-x)} < 0$ for $x \in (-1,0)$ and $f''(x) \geq 0$ for $x \in [0,1)$. Since $f(0) = 0$, we conclude $f(x) > 0$ for $0 < x < 1$ and $f(x) < 0$ for $-1 < x < 0$.

Problem 3. Let the radius of C_n be r_n. Let $A_n = (x_n, 0)$. By Pythagoras' theorem, we get $A_nA_{n-1} = 2\sqrt{r_nr_{n-1}}$, $A_nA_{n-2} = 2\sqrt{r_nr_{n-2}}$ and $A_{n-1}A_{n-2} = 2\sqrt{r_{n-1}r_{n-2}}$. Since $A_{n-1}A_{n-2} = A_nA_{n-1} + A_nA_{n-2}$, we obtain $\frac{1}{\sqrt{r_n}} = \frac{1}{\sqrt{r_{n-1}}} + \frac{1}{\sqrt{r_{n-2}}}$.

If we let $q_n = \frac{1}{\sqrt{2r_n}}$, then $q_n = q_{n-1} + q_{n-2}$ and $q_0 = q_1 = 1$. So q_n is the Fibonacci sequence. Since $A_{n-1}A_n : A_nA_{n-2} = \sqrt{r_{n-1}} : \sqrt{r_{n-2}}$, we also have

$$x_n = \frac{\sqrt{r_{n-2}}x_{n-1} + \sqrt{r_{n-1}}x_{n-2}}{\sqrt{r_{n-1}} + \sqrt{r_{n-2}}} = \frac{q_{n-1}x_{n-1} + q_{n-2}x_{n-2}}{q_{n-1} + q_{n-2}}.$$

In other words, $q_nx_n = q_{n-1}x_{n-1} + q_{n-2}x_{n-2}$. Hence, if we let $p_n = q_nx_n$, then we have $p_n = p_{n-1} + p_{n-2}$. Since $p_0 = q_0x_0 = 0$ and $p_1 = q_1x_1 = 1$, p_n is again the Fibonacci sequence with one term deleted. So $p_n = q_{n-1}$. Therefore,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} \frac{q_{n-1}}{q_n} = \frac{\sqrt{5}-1}{2}.$$

Problem 4. Let $S_0 = \emptyset$ and $S_1, S_2, \ldots, S_{2^n-1}$ be the $2^n - 1$ nonempty distinct subsets of $\{a_1, a_2, \ldots, a_n\}$, where all $a_i > 0$. Let $F(S_i)$ denote the sum of the elements of S_i with $F(S_0) = 0$. By the pigeonhole principle, there exists distinct i, j such that $F(S_i) \equiv F(S_j) \pmod{m}$. Let $S = (S_i \setminus S_j) \cup \{-x : x \in S_j \setminus S_i\}$. Then all a_k and $-a_k$ cannot both be in S and $F(S) \equiv 0 \pmod{m}$.

Problem 5. <u>*Solution 1.*</u> For $t \in [0,1]$, let $u(t) = \left(\int_0^t f(x)\,dx\right)^2$ and $v(t) = \int_0^t f^3(x)\,dx$. For $x \in (0,1)$, $f(0) = 0$ and $f'(x) > 0$ imply $f(x) > 0$. Applying the generalized mean-value theorem twice, there exist $\theta_0, \theta_1 > 0$ such that

$$\frac{u(1)}{v(1)} = \frac{u(1)-u(0)}{v(1)-v(0)} = \frac{u'(\theta_0)-0}{v'(\theta_0)-0} = \frac{2\int_0^{\theta_0} f(t)\,dt - 0}{f^2(\theta_0)-0} = \frac{2f(\theta_1)}{2f(\theta_1)f'(\theta_1)} = \frac{1}{f'(\theta_1)} \geq 1.$$

<u>*Solution 2.*</u> For all $t \in [0,1]$, define $F(t) = \left(\int_0^t f(x)\,dx\right)^2 - \int_0^t f^3(x)\,dx$. All we need to show is $F(1) \geq 0$. Now $F(0) = 0$. For $t \in (0,1)$,

$$F'(t) = 2f(t)\int_0^t f(x)\,dx - f^3(t) = f(t)H(t),$$

where $H(t) = 2\int_0^t f(x)\,dx - f^2(t)$. Then $H(0) = 0$ and $H'(t) = 2f(t) - 2f(t)f'(t) = 2f(t)(1-f'(t)) \geq 0$. So $H(t) \geq H(0) = 0$ for all $t \in [0,1]$. Since $f(0) = 0$ and $f'(t) > 0$, so for all $t \in (0,1]$, $f(t) > f(0) = 0$. Then $F'(t) \geq 0$. Therefore, $F(1) \geq F(0) = 0$.

Problem 6. We have $f'(x) = \dfrac{(36n - x)x}{2^5 \cdot 3^2 \cdot n^2} > 0$ for $0 < x < 36n$. Hence, f is strictly increasing on $[0, 36n]$. So

$$0 = [f(0)] \le [f(1)] \le [f(2)] \le \cdots \le [f(36n)] = 27n.$$

Note that $f'(x) - 1 = -\dfrac{(x - 12n)(x - 24n)}{2^5 \cdot 3^2 \cdot n^2}$. Hence, $f'(x) \le 1$ for $x \in [0, 12n] \cup [24n, 36n]$ and $f'(x) > 1$ for $x \in (12n, 24n)$. Now $f(12n) = 7n$ and $f(24n) = 20n$. For $[0, 12n]$ or $[24n, 36n]$, by the mean value theorem, $f(k+1) - f(k) = f'(c) \le 1$ for some $c \in (k, k+1)$. This means $0 < f(k+1) - f(k) \le 1$. Hence, $[f(k+1)] = [f(k)]$ or $[f(k)] + 1$. Therefore, the range covers all the integers from 0 to $7n$ and from $20n$ to $27n$. On $(12n, 24n)$, by the mean value theorem as above, we conclude that $f(k+1) - f(k) > 1$. Hence,

$$[f(12n)] < [f(12n+1)] < \cdots < [f(24n)].$$

Excluding the endpoints, there are $12n - 1$ distinct values. Therefore, we conclude that there are in total $(7n + 1) + (12n - 1) + (7n + 1) = 26n + 1$ distinct values.