Seventh HKUST Undergraduate Math Competition – Senior Level

April 27, 2019

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1. Prove that there does not exist a continuous function $f : [0, \pi] \to \mathbb{R}$ such that

$$\int_0^\pi |f(x) - \sin x|^2 \, dx \le \frac{3}{4} \quad \text{and} \quad \int_0^\pi |f(x) - \cos x|^2 \, dx \le \frac{3}{4}.$$

Problem 2. Let $T = \{a_1, -a_1, a_2, -a_2, \ldots, a_n, -a_n\}$ be a set of $2n$ distinct integers. Let $1 \le m < 2^n$. Prove that there exists a nonempty subset S of T such that for each $i = 1, 2, \ldots, n$, the integers a_i and $-a_i$ are not both in S and the sum of all elements of S is divisible by m.

Problem 3. Let k be an integer greater than 1. Let G be a finite group of order n. Prove that k and n are relatively prime if and only if every element of G is the k-th power of some element in G.

(Here a *finite group G of order n* is a finite set with n distinct elements such that there exists a function from $G \times G$ to G satisfying (1) for every $(a, b) \in G \times G$, it is assigned to a unique element $c \in G$, where c can be denoted by ab, (2) for every $x, y, z \in G$, we have $(xy)z = x(yz)$, (3) there exists an element $1 \in G$ such that for all $w \in G$, we have $w1 = w = 1w$ and (4) for every $v \in G$, there is a unique element in G denoted by v^{-1} such that $vv^{-1} = 1 = v^{-1}v$.)

Problem 4. Determine the value of $\displaystyle\int_{-\infty}^{+\infty} \frac{x(\sin x - 2e \cos x)}{(1 + x^2)^2} \, dx.$

Problem 5. Let $a_n = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \cdots - \dfrac{(-1)^n}{n} - \ln 2$. Prove that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges and find its sum.

Problem 6. Determine the smallest prime number which can be written in each of the forms: $x_1^2 + y_1^2, x_2^2 + 2y_2^2, \cdots, x_{10}^2 + 10y_{10}^2$, where x_i, y_i are integers for $i = 1, 2, \ldots, 10$.

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