Solutions of 2019 UG Math Competition - Senior Level

Problem 1. Assume the opposite is true. We have

$$\begin{aligned}\int_0^\pi |\sin x-\cos x|^2\,dx &= \int_0^\pi (\sin^2 x-2\sin x\cos x+\cos^2 x)\,dx\\ &= \int_0^\pi (1-\sin 2x)\,dx = \Big(x+\frac{\cos 2x}{2}\Big)\Big|_0^\pi = \pi.\end{aligned}$$

and

$$\begin{aligned}\pi = \int_0^\pi |\sin x-\cos x|^2\,dx &\le \int_0^\pi (|\sin x-f(x)|+|f(x)-\cos x|)^2\,dx\\ &\le 2\int_0^\pi |f(x)-\sin x|^2\,dx+2\int_0^\pi |f(x)-\cos x|^2\,dx \le 2\Big(\frac{3}{4}\Big)+2\Big(\frac{3}{4}\Big)=3,\end{aligned}$$

which is a contradiction.

Problem 2. Let $S_0=\emptyset$ and $S_1,S_2,\ldots,S_{2^n-1}$ be the 2^n-1 nonempty distinct subsets of $\{a_1,a_2,\ldots,a_n\}$, where all $a_i>0$. Let $F(S_i)$ denote the sum of the elements of S_i with $F(S_0)=0$. By the pigeonhole principle, there exists distinct i,j such that $F(S_i)\equiv F(S_j) \pmod m$. Let $S=(S_i\setminus S_j)\cup\{-x : x\in S_j\setminus S_i\}$. Then all a_k and $-a_k$ cannot both be in S and $F(S)\equiv 0 \pmod m$.

Problem 3. Suppose $\gcd(k,n)=1$. If $a\in G$ is of order m, then $m|n$ by Lagrange's theorem. Then $kx\equiv 1 \pmod m$ has a solution since $\gcd(k,m)=\gcd(k,n)=1$. So $(a^x)^k=1$.

Suppose $\gcd(k,n)>1$. Choose a prime p such that $p|\gcd(k,n)$. By Cauchy's theorem, there exists $b\in G$ with $b^p=1$, then $b^k=1$. For every element in G to be a k-th power, it is necessary that the k-th powers of the n elements in G be distinct. Since $b^k=1=1^k$, this is impossible.

Problem 4. Let $S_0=\emptyset$ and $S_1,S_2,\ldots,S_{2^n-1}$ be the 2^n-1 nonempty distinct subsets of $\{a_1,a_2,\ldots,a_n\}$, where all $a_i>0$. Let $F(S_i)$ denote the sum of the elements of S_i with $F(S_0)=0$. By the pigeonhole principle, there exists distinct i,j such that $F(S_i)\equiv F(S_j) \pmod m$. Let $S=(S_i\setminus S_j)\cup\{-x : x\in S_j\setminus S_i\}$. Then all a_k and $-a_k$ cannot both be in S and $F(S)\equiv 0 \pmod m$.

Problem 5. Let $f(z)=ze^{iz}/(1+z^2)^2$. For $R>0$, consider the contour going from $-R$ to R on the x-axis followed by the upper semicircle C_R with the center at 0 and radius R. By the residue theorem,

$$\int_{-R}^{R}\frac{xe^{ix}\,dx}{(1+x^2)^2}+\int_{C_R}\frac{ze^{iz}\,dz}{(1+z^2)^2}=2\pi i\,\mathrm{Res}\Big(\frac{ze^{iz}}{(1+z^2)^2},i\Big)=2\pi i\frac{d}{dz}\Big(\frac{ze^{iz}}{(1+z^2)^2}\Big)\Big|_{z=i}=\frac{\pi i}{2e}.$$

By Jordan's inequality, let $h(z)=\dfrac{z}{(1+z^2)^2}$, then

$$\Big|\int_{C_R}h(z)e^{iz}\,dz\Big|\le\int_{C_R}|h(z)e^{iz}||dz|\le\frac{R}{(R^2-1)^2}\int_0^\pi e^{-R\sin\theta}R\,d\theta\le\frac{R^2}{(R^2-1)^2}\frac{\pi}{R}\to 0.$$

Then we have

$$\int_{-\infty}^{+\infty}\frac{x(\sin x-2e\cos x)}{(1+x^2)^2}\,dx=\mathrm{Im}\int_{-\infty}^{+\infty}\frac{xe^{ix}}{(1+x^2)^2}\,dx-2e\,\mathrm{Re}\int_{-\infty}^{+\infty}\frac{xe^{ix}}{(1+x^2)^2}\,dx=\frac{\pi}{2e}.$$

Problem 6. If $x^2+ry^2=p$, then $x^2\equiv -ry^2 \pmod p$. So $-r$ is a quadratic residue of p. Hence, $\Big(\dfrac{-r}{p}\Big)=1$ for $1\le r\le 10$. It is sufficient to have $-1,2,3,5$ and 7 are quadratic residues of p. This follows from having $p\equiv 1 \pmod{2^3}$, $p\equiv 1 \pmod 3$, $p\equiv 1$ or $-1 \pmod 5$ and $p\equiv 1,2$ or $4 \pmod 7$. Then p must satisfy $p\equiv 1^2,11^2,13^2,17^2,19^2$ or $23^2 \pmod{840}$. The smallest such p is 1009. We have

$$\begin{aligned}1009 &= 15^2+28^2=19^2+2\cdot 18^2=31^2+3\cdot 4^2=15^2+4\cdot 14^2=17^2+5\cdot 12^2\\ &=25^2+6\cdot 8^2=1^2+7\cdot 12^2=19^2+8\cdot 9^2=28^2+9\cdot 5^2=3^2+10\cdot 10^2.\end{aligned}$$