2023 HKUST Undergraduate Math Competition – Junior Level

No Calculators are allowed. For each problem, provide complete details of your solution.

Problem 1. (15 points) Let P be a convex polygon, prove that there exists a straight line that divides P into two polygons with equal area and equal perimeter. (For example, if P is a square, any line passing through the center of P is such a line).

Problem 2.(15 points) Show that the improper integral

$$\lim_{B\to\infty}\int_0^B \sin x \sin x^2 \, dx$$

hint: using the integration by parts.

Problem 3.(15 points) Let p, q be distinct prime numbers. Prove that

$$\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \cdots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor = \frac{(p-1)(q-1)}{2}.$$

Here $\lfloor x \rfloor$ denote the largest positive integer not exceeding x for any real number x. For example, $\lfloor \pi \rfloor = \lfloor 3.14 \rfloor = 3$.

Problem 4. (15 points) Suppose that a sequence $b_1, b_2, b_3, \cdots$ satisfies $0 < b_n \leq b_{2n} + b_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} b_n$ diverges.

Problem 5. (15 points) Let $f, g : \mathbb{R} \to \mathbb{R}$ be periodic functions such that $\lim_{x\to\infty}(f(x) - g(x)) = 0$. Prove that $f = g$.

Problem 6. (15 points) (1) Let $a > 2$, suppose x_0 is a root of $x^2 - ax + 1 = 0$,

prove that $y_0 = \sqrt{x_0}$ is a root of $y^2 - \sqrt{a+2}y + 1 = 0$.
(2) Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$ where a, b, c, d are integers.

Problem 7.(10 points) Prove that every $n \times n$ invertible real matrix A can be written as $A = BK$, where B is an $n \times n$ positive definite matrix and K is an $n \times n$ orthogonal matrix , i.e., $K^T K = I_n$.