2023 HKUST Undergraduate Math Competition – Senior Level

No Calculators are allowed. For each problem, provide complete details of your solution.

Problem 1. (15 points) Let $\mathbb{C}^*$ be the complex plane with 0 removed, let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a holomorphic map that is a bijection. Show that there is a number $a \in \mathbb{C}^*$ such that either $f(z) = az$ or $f(z) = az^{-1}$.

Problem 2. (15 points) Let V be the space of complex valued continuous functions $f(x)$ on $\mathbb{R}$ satisfying the periodicity condition $f(x+1) = f(x)$. For any positive integer n, we define n-th Hecke operator T_n on a continuous function $f(x)$ by

$$(T_n f)(x) = \sum_{j=0}^{n-1} f(\frac{1}{n}x + \frac{j}{n}).$$

(1) Prove that if $f(x) \in V$, then so is $(T_n f)(x)$. So we have an linear operator $T_n : V \to V$.
(2) Prove that $T_m T_n = T_{mn}$.
(3) Can you find two common eigenfunctions for T_n $(n = 1, 2, \ldots)$? hint: consider the functions $e^{2\pi imx}$ first.

Problem 3. (15 points) Let n be a positive integer, and let $S(n)$ denote the sum of its decimal digits. For example, $S(2357) = 2+3+5+7 = 17$. Prove the following:

(1) $9|S(n) - n$;

(2) $S(n_1 + n_2) \leq S(n_1) + S(n_2)$;

(3) $S(n_1 n_2) \leq \min\{n_1 S(n_2), n_2 S(n_1)\}$;

(4) $S(n_1n_2) \le S(n_1)S(n_2)$.

(5) Suppose n is a positive integer such that in its decimal expansion, each digit (except the first digit) is greater than the digit to its left. What is $S(9n)$, and why?

Here n, n_1 and n_2 denote any positive integers.

Problem 4. (15 points) Let R be the ring of analytic functions on the complex plane, is R an integral domain? why?

Problem 5. (15 points) Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum_{i=1}^{n} \frac{1}{1+x_i} = 1$. Prove that $\sum_{i=1}^{n} \sqrt{x_i} \ge (n-1) \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}}$.

Problem 6. (15 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function such that $f(0) = 1$, $f'(0) = 0$, and for all $x \in [0, \infty)$,

$$f''(x) - 5f'(x) + 6f(x) \ge 0.$$

Show that for all $x \in [0, \infty)$,

$$f(x) \ge 3e^{2x} - 2e^{3x}.$$

Problem 7. (10 points) Let A be an $n \times n$ symmetric real matrix with (i,j)-entry $a_{ij} = a_{ji}$, A defines a function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = x^T A x = \sum_{i,j=1}^{n} a_{ij} x_i x_j$. Suppose $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$ satisfies the conditions that
(1) c is a unit vector, i.e, $c_1^2 + \cdots + c_n^2 = 1$
(2) $f(c) \ge f(v)$ for all unit vector $v \in \mathbb{R}^n$. Prove that c is an eigenvector of A and the eigenvalue of c is the largest eigenvalue of A.