Solutions to 2023 HKUST Math Competition – Senior Level

Problem 1. (15 points) Let $\mathbb{C}^*$ be the complex plane with 0 removed, let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a holomorphic map that is a bijection. Show that there is a number $a \in \mathbb{C}^*$ such that either $f(z) = az$ or $f(z) = az^{-1}$.

Proof If f has essential singularity By Great Picard theorem, in arbitrary neighborhood U of 0 the set $f(U - \{0\})$ exhausts all elements in $\mathbb{C}^*$ except one value. This violates the condition f is a bijection. Therefore f has pole or removable singularity at 0. Similarly f has pole or removable singularity regarded as a function near ∞. This implies f extends to a holomorphic map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree one, and thus must be an automorphism of $\mathbb{P}^1$, which we know is of form $f(z) = \frac{az+b}{cz+d}$ for some constant $a, b, c, d \in \mathbb{C}$. Compare to the condition of f on $\mathbb{C}$, we then have $f(0) = 0, f(\infty) = \infty$ or $f(0) = \infty, f(\infty) = 0$. In first case one gets $b = 0 = c$, and then $f(z) = (a/d)z$. In the second case one gets $d = 0 = a$, thus $f(z) = (b/c)z^{-1}$. This proves the claim.

Problem 2. (15 points) Let V be the space of complex valued continuous functions $f(x)$ on $\mathbb{R}$ satisfying the periodicity condition $f(x + 1) = f(x)$. For any positive integer n, we define n-th Hecke operator T_n on a continuous function $f(x)$ by

$$(T_n f)(x) = \sum_{j=0}^{n-1} f(\frac{1}{n}x + \frac{j}{n}).$$

(1) Prove that if $f(x) \in V$, then so is $(T_n f)(x)$. So we have an linear operator $T_n : V \to V$.

(2) Prove that $T_m T_n = T_{mn}$.

(3) Can you find two common eigenfunctions for T_n $(n = 1, 2, \dots)$? hint: consider the functions $e^{2\pi imx}$ first.

Answer: (1)

$$\begin{aligned}(T_n f)(x+1) &= \sum_{j=0}^{n-1} f(\frac{1}{n}(x+1)+\frac{j}{n}) \\ &= \sum_{j=0}^{n-1} f(\frac{1}{n}x+\frac{j+1}{n}) \\ &= \sum_{j=1}^{n-1} f(\frac{1}{n}x+\frac{j}{n}) + f(\frac{1}{n}x+1) \\ &= \sum_{j=1}^{n-1} f(\frac{1}{n}x+\frac{j}{n}) + f(\frac{1}{n}x) \\ &= \sum_{j=0}^{n-1} f(\frac{1}{n}x+\frac{j}{n}) = (T_n f)(x)\end{aligned}$$

This proves $T_n f \in V$.

(2)

$$(T_m T_n f)(x) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} f(\frac{1}{mn}x+\frac{i}{mn}+\frac{j}{n}) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} f(\frac{1}{mn}x+\frac{i+mj}{mn})$$

Note that when i runs through 0 to $m-1$ and j runs through 0 to $n-1$, $i+mj$ runs through 0 to $mn-1$, so $(T_m T_n f)(x) = (T_{mn} f)(x)$.

(3) The constant function 1 is an common eigenfunction $T_n 1 = n = n \cdot 1$. A less obvious common eigenfunction is $f_s(x) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}$, where s is a complex number with $\operatorname{re} s > 1$ (this condition is for the convergence). Then $T_n f_s(x) = n^{1-s} f_s(x)$.

Problem 3. (15 points) Let n be a positive integer, and let $S(n)$ denote the sum of its decimal digits. For example, $S(2357) = 2+3+5+7 = 17$. Prove the following:

(1) $9 | S(n) - n$;

(2) $S(n_1 + n_2) \le S(n_1) + S(n_2)$;

(3) $S(n_1 n_2) \le \min\{n_1 S(n_2), n_2 S(n_1)\}$;

(4) $S(n_1 n_2) \le S(n_1)S(n_2)$.

(5) Suppose n is a positive integer such that in its decimal expansion, each digit (except the first digit) is greater than the digit to its left. What is $S(9n)$, and why?

Here n, n_1 and n_2 denote any positive integers.

Proof

(1). Let $n = \overline{a_k a_{k-1} \cdots a_0}$. Then $S(n) = a_k + a_{k-1} + \cdots + a_0$ and $n = a_k * 10^k + a_{k-1} * 10^{k-1} + \cdots + a_0$. Since $10 \equiv 1 \pmod 9$, obviously

$$n \equiv a_k + a_{k-1} + \cdots + a_0 \pmod 9 \equiv S(n) \pmod 9.$$

(2). Suppose $n_1 = \overline{a_k a_{k-1} \cdots a_0}, n_2 = \overline{b_h b_{h-1} \cdots b_0}$, and $n_1 + n_2 = \overline{c_s c_{s-1} \cdots c_0}$. Let t be least such that $a_i + b_i < 10$ for all $i < t$. Then $a_t + b_t \ge 10$ and hence $c_t = a_t + b_t - 10$ and $c_{t+1} \le a_{t+1} + b_{t+1} + 1$. We obtain

$$\sum_{i=0}^{t+1} c_i \le \sum_{i=0}^{t+1} a_i + \sum_{i=0}^{t+1} b_i.$$

Continuing this procedure, the conclusion follows.

(3). Applying (2) n_1 times, we obtain

$$\begin{aligned} S(n_1 n_2) &= S(n_2 + (n_1 - 1) * n_2) \le S(n_2) + S((n_1 - 1)n_2) \\ &\le \cdots \le S(n_2) + S(n_2) + \cdots + S(n_2) = n_1 S(n_2). \end{aligned}$$

By symmetry, we also have $S(n_1 n_2) \le n_2 S(n_1)$.

(4).

$$\begin{aligned} S(n_1 n_2) &= S(n_1 \sum_{i=0}^{h} b_i * 10^i) = S(\sum_{i=0}^{h} n_1 b_i * 10^i) \le \sum_{i=0}^{h} S(n_1 * b_i) \\ &\le \sum_{i=0} b_i S(n_1) = S(n_1)S(n_2). \end{aligned}$$

(5). Write $n = \overline{a_k a_{k-1} \cdots a_0}$. By performing the subtraction

$$\begin{array}{cccccc} & a_k & a_{k-1} & \ldots & a_1 & a_0 & 0 \\ - & & a_k & \ldots & a_2 & a_1 & a_0 \end{array}$$

we find that the digits of $9n = 10n - n$ are $a_k, a_{k-1} - a_k, \ldots, a_1 - a_2 - 1, 10 - a_0$. These digits sum to $10 - 1 = 9$.

Problem 4. (15 points) Let R be the ring of analytic functions on the complex plane, is R an integral domain? why?

Answer: It is obvious that R is a commutative ring with 1. If $f(z)g(z) = 0$ for some analytic functions $f(z)$ and $g(z)$ on $\mathbb{C}$, then $Z(f) \cup Z(g) = \mathbb{C}$, where $Z(f)$ denotes the set of zeros of $f(z)$, $Z(g)$ has the similar meaning. In particular, one of the sets $Z(f) \cap \{z \,|\, |z| = 1\}$ and $Z(g) \cap \{z \,|\, |z| = 1\}$ must be an infinite set. We may assume $Z(f) \cap \{z \,|\, |z| = 1\}$ is infinite, so the zeros of $f(x)$ has a limit point in $\{z \,|\, |z| = 1\}$, this implies $f(z) = 0$. This proves R has no zero divisor, so it is an integral domain.

Problem 5. (15 points) Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum_{i=1}^n \frac{1}{1+x_i} = 1$. Prove that $\sum_{i=1}^n \sqrt{x_i} \geq (n-1) \sum_{i=1}^n \frac{1}{\sqrt{x_i}}$.

Proof. Let $a_i = \frac{1}{1+x_i}$. Using the condition $\sum_{i=1}^n \frac{1}{1+x_i} = 1$, we see that

$$\sqrt{x_i} = \sqrt{\frac{a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n}{a_i}}$$

It is enough to prove

$$(n-1) \sum_{i=1}^n \sqrt{\frac{a_i}{a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n}} \leq \sum_{i=1}^n \sqrt{\frac{a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n}{a_i}}.$$

By using the Cauchy-Schwartz inequality, we have

$$\sum_{i=1}^{n}\sqrt{\frac{a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_n}{a_i}}$$
$$\geq \sum_{i=1}^{n}\frac{\sqrt{a_1}+\cdots+\sqrt{a_{i-1}}+\sqrt{a_{i+1}}+\cdots+\sqrt{a_n}}{\sqrt{n-1}\sqrt{a_i}}$$
$$=\sum_{i=1}^{n}\frac{\sqrt{a_i}}{\sqrt{n-1}}\left(\frac{1}{\sqrt{a_1}}+\cdots+\frac{1}{\sqrt{a_{i-1}}}+\frac{1}{\sqrt{a_{i+1}}}+\cdots+\frac{1}{\sqrt{a_n}}\right)=B$$

Using the inequality

$$x_1+\cdots+x_{n-1}\geq (n-1)^2\frac{1}{x_1^{-1}+\cdots+x_{n-1}^{-1}}$$

for each of the summands in (B) above and using the Cauchy-Schwartz inequality, we have

$$B\geq \sum_{i=1}^{n}(n-1)\sqrt{\frac{a_i}{a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_n}}$$

Problem 6. (15 points) Let $f:\mathbb{R}\to\mathbb{R}$ be a twice-differentiable function such that $f(0)=1$, $f'(0)=0$, and for all $x\in[0,\infty)$,

$$f''(x)-5f'(x)+6f(x)\geq 0.$$

Show that for all $x\in[0,\infty)$,

$$f(x)\geq 3e^{2x}-2e^{3x}.$$

Solution: Let $g(x)=f'(x)-2f(x)$. Then the given inequality is equivalent to

$$g'(x)-3g(x)\geq 0, \qquad x\in[0,\infty),$$

and hence,

$$(g(x)e^{-3x})'\geq 0, \qquad x\in[0,\infty).$$

Thus, $g(x)e^{-3x}$ is an increasing function on $[0,\infty)$, which implies that

$$g(x)e^{-3x}\geq g(0)=-2, \qquad x\in[0,\infty),$$

or equivalently,

$$f'(x) - 2f(x) \geq -2e^{3x}, \qquad x \in [0, \infty).$$

As above, we get

$$(f(x)e^{-2x})' \geq -2e^{x}, \qquad x \in [0, \infty),$$

or equivalently,

$$(f(x)e^{-2x} + 2e^{x})' \geq 0, \qquad x \in [-, \infty).$$

This implies that

$$f(x)e^{-2x} + 2e^{x} \geq f(0) + 2 = 3, \qquad x \in [0, \infty)$$

which means

$$f(x) \geq 3e^{2x} - 2e^{3x}, \qquad x \in [0, \infty).$$

Problem 7. (10 points) Let A be an $n \times n$ symmetric real matrix with (i, j)-entry $a_{ij} = a_{ji}$, A defines a function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = x^T Ax = \sum_{i,j=1}^{n} a_{ij}x_ix_j$. Suppose $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$ satisfies the conditions that
(1) c is a unit vector, i.e, $c_1^2 + \cdots + c_n^2 = 1$
(2) $f(c) \geq f(v)$ for all unit vector $v \in \mathbb{R}^n$. Prove that c is an eigenvector of A and the eigenvalue of c is the largest eigenvalue of A.

Proof 1. Using the Lagrangian multiplier method, set

$$F(x_1, \ldots, x_n, \lambda) = f(x) + \lambda(x_1^2 + \cdots + x_n^2 - 1)$$

we see that the vector c and some λ_0 satisfies the condition that

$$\frac{\partial F}{\partial x_i}(c_1, \ldots, c_n, \lambda_0) = 0$$

for $i = 1, \ldots, n$. Which equivalent to $Ac = \lambda_0 c$. This proves c is an eigenvector with eigenvalue λ_0. If λ is another eigenvalue of A, let v be a unit eigenvector with eigenvalue λ, then suing $f(v) = v^T A v = \lambda \leq f(c) = \lambda_0$, we prove $\lambda \leq \lambda_0$

Sketch of Proof 2. Write A as $A = K^T D K$ for some orthogonal matrix K and diagonal matrix D, since the unit ball $\{x \in \mathbb{R}^n \mid |x| = 1\}$ is invariant under the transformation $x \mapsto Kx$, the problem reduces to the case $A = D$, where the solution is given by a direct computation.