9th HKUST Undergraduate Math Competition – Junior Level
April 27th, 2024

In the following, $\mathbb{R}$ denotes real numbers and $n \in \mathbb{N}$ denotes positive integers.

Problem 1. Suppose f is a continuous function on $\mathbb{R}$ and $f(f(x)) = x$. Prove that there exists $c \in \mathbb{R}$ such that

$$f(c) = c.$$

Problem 2. Let $\mathbf{A}, \mathbf{B}$ be $n \times n$ matrices with real entries. Assume $\mathbf{A}, \mathbf{B}$ and $\mathbf{A} + \mathbf{B}$ are invertible, and moreover

$$\mathbf{A}^{-1} + \mathbf{B}^{-1} = (\mathbf{A} + \mathbf{B})^{-1}.$$

Show that

$$\det \mathbf{A} = \det \mathbf{B}.$$

Problem 3. Let $a_1, a_2, ..., a_{10}$ be integers with $1 \le a_i \le 25$ and $1 \le i \le 10$. Prove that there exists integers $m_1, m_2, ..., m_{10}$ not all zero, such that

$$\prod_{i=1}^{10} a_i^{m_i} = 1.$$

Problem 4. Let (x, y) denote the coordinate of a randomly chosen point on a unit circle, uniformly over the area of the circle. Calculate the expected value

$$\mathbb{E}(\max(|x|, |y|)).$$

Problem 5. Let $f_n(x)$ be a sequence of monotone decreasing functions on $[0, 1]$ with $0 \le f_n(x) \le 1$. Define recursively

$$A_1 := \int_0^1 f_1(x)dx$$

$$A_n := \int_0^{A_{n-1}} f_n(x)dx, \qquad n \ge 2.$$

Prove that

$$\int_0^1 f_1(x)f_2(x)\cdots f_{2024}(x)dx \le A_{2024}.$$

Problem 6. Let C_0 and C_1 be two unit circles centered at $(0, 1)$ and $(2, 1)$ respectively. Let C_{n+1} be the circle (different from C_{n-2} if $n \ge 2$) that is touching C_n, C_{n-1}, and touching the x-axis at $(x_n, 0)$. Find

$$\lim_{n\to\infty} x_n.$$

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