9th HKUST Undergraduate Math Competition – Junior Level

Suggested Solutions

Problem 1. Take any $a \in \mathbb{R}$, if $f(a) = a$, we are done. If $f(a) = b \neq a$, then $f(f(a)) = f(b)$, so $f(b) = a$. Now consider the function

$$F(x) = f(x) - x,$$

we have

$$F(a) = f(a) - a = b - a \neq 0, \quad F(b) = f(b) - b = a - b \neq 0.$$

$F(a)$ and $F(b)$ have different signs. By the intermediate value theorem, there is exists $b < c < a$ such that $F(c) = 0$, so $f(c) = c$.

Problem 2. We have

$$I = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B})^{-1} = (\mathbf{A} + \mathbf{B})(\mathbf{A}^{-1} + \mathbf{B}^{-1}) = 2I + \mathbf{AB}^{-1} + \mathbf{BA}^{-1}$$

and hence

$$\mathbf{AB}^{-1} + \mathbf{BA}^{-1} + \mathbf{I} = 0.$$

Let $\mathbf{X} = \mathbf{AB}^{-1}$ then $\mathbf{X}^{-1} = \mathbf{BA}^{-1}$ and the equation above becomes

$$\mathbf{X} + \mathbf{X}^{-1} + \mathbf{I} = 0$$

which implies, by multiplying both sides with $(\mathbf{X} - \mathbf{I})\mathbf{X}$, that

$$\mathbf{X}^3 - \mathbf{I} = 0,$$

or equivalently, $\mathbf{X}^3 = \mathbf{I}$. In particular, $(\det\mathbf{X})^3 = 1$. But since $\det\mathbf{X}$ is a real number, $\det\mathbf{X} = 1$, which means $\det\mathbf{A}(\det\mathbf{B})^{-1} = \det(\mathbf{AB}^{-1}) = 1$ and hence, $\det\mathbf{A} = \det\mathbf{B}$.

Problem 3. There are nine prime numbers ≤ 25: $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13, p_7 = 17, p_8 = 19, p_9 = 23$. By unique factorization, for each $1 \leq a \leq 25$ there is an integer sequence $v(a) = (v_j(a))_{j=1}^9$ with

$$a = \prod_{j=1}^{9} p_j^{v_j(a)}.$$

The 10 sequences $v(a_i) \in \mathbb{Q}^9$ must be linearly dependent, so

$$\sum_{i=1}^{10} m_i v_j(a_i) = 0$$

for all j, for some rational numbers m_i which are not all 0. Multiplying by a common multiple of the denominators, we can assume that the m_i's are integers, so

$$\prod_{i=1}^{10} a_i^{m_i} = \prod_{j=1}^{9} p_j^{\sum_{i=1}^{10} m_i v_j(a_i)} = 1,$$

as required.

Problem 4. Using polar coordinates, we just need to calculate

$$\begin{aligned}
\mathbb{E}(\max|x|,|y|) &= \frac{\displaystyle\int_{D^2} \max(|x|,|y|)dxdy}{\text{Area of circle}} \\
&= \frac{\displaystyle\int_0^1 \int_0^{2\pi} \max(|r\cos\theta|,|r\sin\theta|) r dr d\theta}{\pi} \\
\text{(symmetry)} &= \frac{8}{\pi}\int_0^1 \int_0^{\frac{\pi}{4}} r^2 \cos\theta d\theta \\
&= \frac{4\sqrt{2}}{3\pi}
\end{aligned}$$

Problem 5. We first prove the case for $n = 2$. Rename $f = f_1$ and $g = f_2$.

Let $\lambda = \displaystyle\int_0^1 f(x)dx$. Note that $0 \leq \lambda \leq 1$. We have

$$\begin{aligned}
&\int_0^\lambda g(x)dx - \int_0^1 f(x)g(x)dx \\
&= \int_0^\lambda g(x)(1-f(x))dx - \int_\lambda^1 f(x)g(x)dx \\
&\geq g(\lambda)\left(\lambda - \int_0^\lambda f(x)dx\right) - \int_\lambda^1 f(x)g(x)dx \\
&= g(\lambda)\left(\int_0^1 f(x)dx - \int_0^\lambda f(x)dx\right) - \int_\lambda^1 f(x)g(x)dx \\
&= g(\lambda)\int_\lambda^1 f(x)dx - \int_\lambda^1 f(x)g(x)dx \\
&= \int_\lambda^1 f(x)(g(\lambda) - g(x))dx \\
&\geq 0
\end{aligned}$$

Hence

$$\int_0^1 f(x)g(x) \leq \int_0^{\int_0^1 f(x)dx} g(x)dx$$

and the statement follows easily by induction on n by letting $f(x) := f_1(x)\cdots f_{n-1}(x)$ and $g(x) := f_n(x)$.

Problem 6. Let $A_n = (x_n, 0)$. Also let $x_{-1} = 0$ and $A_{-1} = (0,0)$. Then easily see that

$$A_n A_{n-1} = 2\sqrt{r_n r_{n-1}}$$
$$A_n A_{n-2} = 2\sqrt{r_n r_{n-2}}$$
$$A_{n-1} A_{n-2} = 2\sqrt{r_{n-1} r_{n-2}}$$

Since $A_{n-1}A_{n-2} = A_n A_{n-1} + A_n A_{n-2}$ we obtain

$$\frac{1}{\sqrt{r_n}} = \frac{1}{\sqrt{r_{n-1}}} + \frac{1}{\sqrt{r_{n-2}}}$$

If we let $q_n = \frac{1}{\sqrt{2r_n}}$, then $q_n = q_{n-1} + q_{n-2}$ and $q_{-1} = q_0 = \frac{1}{\sqrt{2}}$, hence $q_n = \frac{1}{\sqrt{2}} F_{n+1}$ where F_n is the Fibonacci sequence.

Since $A_{n-1}A_n : A_nA_{n-2} = \sqrt{r_{n-1}} : \sqrt{r_{n-2}}$, we also have

$$x_n = \frac{\sqrt{r_{n-2}}x_{n-1} + \sqrt{r_{n-1}}x_{n-2}}{\sqrt{r_{n-1}} + \sqrt{r_{n-2}}} = \frac{q_{n-1}x_{n-1} + q_{n-2}x_{n-2}}{q_{n-1} + q_{n-2}}$$

in other words

$$q_n x_n = q_{n-1}x_{n-1} + q_{n-2}x_{n-2}.$$

Hence if we let $p_n = q_n x_n$, we have

$$p_n = p_{n-1} + p_{n-2}$$

since $p_{-1} = q_{-1}x_{-1} = 0$, $p_0 = q_0 x_0 = \sqrt{2}$, hence $p_n = \sqrt{2}F_n$ is again Fibonacci sequence. Therefore

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} \frac{\sqrt{2}F_n}{\frac{1}{\sqrt{2}}F_{n+1}} = \sqrt{5} - 1.$$

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