9th HKUST Undergraduate Math Competition – Senior Level
April 27th, 2024

In the following, $\log x$ denotes the natural logarithm.

Problem 1. Determine the last two digits of

$$23^{23^{23^{23}}}$$

in the decimal system.

Problem 2. Let p be a prime number, and $SL_2(\mathbb{F}_p)$ be the group of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_p$ and determinant 1. Show that there is no injective group homomorphism

$$\phi : SL_2(\mathbb{F}_p) \hookrightarrow S_p$$

where S_p is the symmetric group of p elements.

Problem 3. Let $\mathcal{C}[0,1]$ be the ring of continuous real-valued function on the closed interval $[0,1]$. For each $c \in [0,1]$, let

$$\mathcal{I}_c := \{f \in \mathcal{C}[0,1] \mid f(c) = 0\}.$$

Prove that $\mathcal{I}_c$ is a maximal ideal of $\mathcal{C}[0,1]$, and every maximal ideal of $\mathcal{C}[0,1]$ is of the form $\mathcal{I}_c$ for some $c \in [0,1]$.

Problem 4. Let $\{q_i\}$ be an enumeration of $\mathbb{Q}$ and define a new metric on $\mathbb{R}$ by

$$d(x,y) := |x-y| + \sum_{i=1}^{\infty} 2^{-i} \inf\left(1, \left|\max_{j\le i} \frac{1}{|x-q_j|} - \max_{j\le i} \frac{1}{|y-q_j|}\right|\right).$$

Show that the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is complete with respect to the metric d.
(You are not required to show that d is a metric.)

Problem 5. Let X, Y be two independent random variables on $\mathbb{R}$ satisfying the Pareto distribution with parameter 2, i.e. with probability density function $f(x) = \dfrac{\mathbf{1}_{\{x>1\}}}{x^2}$. We denote

$$(W, Z) = \left(\log X, 1 + \frac{\log Y}{\log X}\right).$$

Determine the probability distribution of W and Z.

Problem 6. Using $\mathrm{Li}_2'(x) = -\dfrac{\log(1-x)}{x}$ or otherwise, evaluate

$$\int_0^1 \frac{\log x \log^2(1-x)}{x} dx.$$

Here $\mathrm{Li}_2(x) := \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ denotes the Euler's dilogarithm function.

–End of Paper–