9th HKUST Undergraduate Math Competition – Senior Level

Suggested Solutions

Problem 1. By Euler's Theorem, $a^{\phi(n)} \equiv 1 \pmod{n}$.

Note that $\phi(100) = 40$, hence we need to know

$$23^{23^{23}} \pmod{40}$$

Since $\phi(40) = 16$, we need to know

$$23^{23} \equiv 7^7 \pmod{16}$$

By direct computation, we get

$$7^7 \equiv 49 \times 49 \times 49 \times 7 \equiv 1 \times 1 \times 1 \times 7 \equiv 7 \pmod{16},$$

and

$$23^{23^{23}} \equiv 23^7 \equiv 23^2 \times 23^2 \times 23^2 \times 7 \equiv 9 \times 9 \times 9 \times 7 \equiv 7 \pmod{40}$$

hence again by direct computation

$$23^{23^{23^{23}}} \equiv 23^7 \equiv 23^2 \times 23^2 \times 23^2 \times 23 \equiv 29 \times 29 \times 29 \times 23 \equiv 41 \times 67 \equiv 47 \pmod{100}.$$

Therefore the last two digits are 47.

Problem 2. When $p = 2$, we see that $\mathrm{SL}_2(\mathbb{F}_p)$ has more than $2 = |S_p|$ elements and hence, such a φ cannot exist.

When $p > 2$, consider

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Observe that A and B have order p and 2, respectively and that they commute. Thus, AB has order $2p$. But there is no permutation in S_p of order $2p$: only p-cycles have order divisible by p, and their order is exactly p.

Problem 3. Let $T_c : f(x) \mapsto f(c)$ be the evaluation map. It is clear that T_c is surjective with kernel $\mathcal{I}_c$, hence the quotient ring $\mathcal{C}[0,1]/\mathcal{I}_c \simeq \mathbb{R}$ by the homomorphism theorem. Since $\mathbb{R}$ is a field, so $\mathcal{I}_c$ is a maximal ideal.

Conversely, suppose $\mathcal{I}$ is a maximal ideal. We prove $\mathcal{I} = \mathcal{I}_c$ by contradiction. Suppose $\mathcal{I} \neq \mathcal{I}_c$ for any c, there exists $f_c \in \mathcal{I}$ such that $f_c \notin \mathcal{I}_c$, so $f_c(c) \neq 0$. so $f_c(x)$ is not 0 everywhere in an open neighborhood of V_c containing c. It is clear that the open sets V_c (as c varies over $[0,1]$) covers $[0,1]$. Because $[0,1]$ is compact, this cover has a finite subcover. That is, there is $c_1, \ldots, c_n$ such that

$$V_{c_1} \cup \cdots \cup V_{c_n} = [0,1].$$

Since $f_{c_i}(x)$ is not 0 at any point in V_{c_i}, so $f_{c_i}(x)^2 > 0$ for all $x \in V_c$. So $F(x) = \sum_{i=1}^{n} f_{c_i}(x)^2 > 0$ for all $x \in [0,1]$. So $\frac{1}{F(x)} \in C[0,1]$. Since $f_{c_i}(x) \in \mathcal{I}$, $F(x) \in \mathcal{I}$, $1 = \frac{1}{F(x)} \cdot F(x) \in \mathcal{I}$, so $\mathcal{I} = C[0,1]$, this contradicts to the assumption $\mathcal{I}$ is a maximal ideal.

(You must use compactness of $[0,1]$ since the statement is wrong on e.g. $\mathcal{C}(0,1)$.)

Problem 4. It is clear that d is a metric by the triangle inequality of the absolute values.

Let (x_n) be a Cauchy sequence in d. Since $|x-y| \leq d(x,y)$, it is also a Cauchy in $\mathbb{R}$.

We claim that (x_n) cannot converge to a rational number: Suppose $x_n \to q_k$. Then for any n, there exists $m > n$ such that $d(x_n, x_m) \geq |x_n - x_m| + 2^{-k}$, a contradiction to (x_n) being Cauchy.

On the other hand, we claim that $d(x,y)$ is equivalent to the Euclidean metric $|x-y|$ on $\mathbb{R} \setminus \mathbb{Q}$: it is clear that the open ball $B_d(p,\epsilon) \subset B_{|\cdot|}(p,\epsilon)$. We show that for any $\epsilon > 0$, there exists $\delta > 0$ such that $(\mathbb{R} \setminus \mathbb{Q}) \cap B_{|\cdot|}(p,\delta) \subset B_d(p,\epsilon)$.

We do a rough estimate as follows. For any $\epsilon > 0$, we split the summation in d into $\sum_{i=1}^{N}$ and $\sum_{i=N}^{\infty}$. Choose N big enough so that the second summation $< \frac{1}{2^{N+1}} < \frac{\epsilon}{3}$.

For the initial terms, if we take $\delta < \frac{\epsilon}{3}$ such that $\delta < \frac{|p-q_i|}{2}$ for all $i \leq N$. Then for $|x-p| < \delta$, and fixed $i \leq N$ both $\max\limits_{j \leq i} \dfrac{1}{|x-q_j|}$ and $\max\limits_{j \leq i} \dfrac{1}{|p-q_j|}$ will be dominated by the same choice of $q_{k_i} \in \mathbb{Q}$ (closest to both x and p and lying on same side) for some $k_i \leq i$, hence the term in the summation will be $= \frac{|x-p|}{|x-q_{k_i}||p-q_{k_i}|} < \frac{\delta}{|x-q_{k_i}||p-q_{k_i}|} < \frac{2\delta}{|p-q_{k_i}|^2}$. Since we have only finitely many such terms with nonzero denominator, the total contribution can be chosen to be $< \frac{\epsilon}{3}$ if δ is small enough.

Therefore if a Cauchy sequence (x_n) does not converge to rational, it must converge to some irrational with respect to d, hence $(\mathbb{R} \setminus \mathbb{Q}, d)$ is complete.

Alternative Solution (by Ji Wenzhou). We show that d and $\|\cdot\|$ are equivalent as follows. As before, $\lim\limits_{n\to\infty} |x_n - x| = 0 \Longrightarrow \lim\limits_{n\to\infty} d(x_n, x) = 0$.

On the other hand, assume $|x_n - x| \to 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. We show that $\lim\limits_{n\to\infty} d(x_n, x) = 0$. Note that the term $\sum 2^{-i} \inf \Big(\cdots \Big)$ in the definition of d is absolutely convergent, so we can interchange limit and summation.

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} |x_n - x| + \sum_{i=1}^{\infty} 2^{-i} \lim_{n\to\infty} \inf \left(1, \left| \max_{j \leq i} \frac{1}{|x_n - q_j|} - \max_{j \leq i} \frac{1}{|x - q_j|} \right| \right)$$

Hence it is enough to argue that for each fixed i,

$$\lim_{n\to\infty} \max_{j \leq i} \frac{1}{|x_n - q_j|} = \max_{j \leq i} \frac{1}{|x - q_j|},$$

which is clear since $\max(\cdots)$ is a continuous function on finitely many terms and the denominator is nonzero for irrational x_n and x.

Problem 5. Let $\phi : \mathbb{R}^2_{>1} \to \mathbb{R}_{>0} \times \mathbb{R}_{>1}$ be given by

$$\phi(x,y) = (w,z) = \left(\log x, 1 + \frac{\log y}{\log x}\right).$$

It is invertible with

$$\begin{cases} w = \log x \\ z = 1 + \log y / \log x \end{cases} \Leftrightarrow \begin{cases} \log x = w \\ \log y = w(z-1) \end{cases} \Leftrightarrow \begin{cases} x = e^w \\ y = e^{w(z-1)} \end{cases}$$

and the Jacobian $|D\phi^{-1}| = \begin{vmatrix} e^w & 0 \\ (z-1)e^{w(z-1)} & we^{w(z-1)} \end{vmatrix} = we^{wz}$.

The joint probability density is then given by

$$f(w,z) = (\log x)(1 + \frac{\log y}{\log x})\frac{\mathbf{1}_{x>1}}{x^2}\frac{\mathbf{1}_{y>1}}{y^2} = \mathbf{1}_{w>0}\mathbf{1}_{z>1}\frac{we^{wz}}{e^{2w}e^{2w(z-1)}}|D\phi^{-1}| = \mathbf{1}_{w>0}\mathbf{1}_{z>1}we^{-wz}.$$

By the formula of marginal density, the density of W is given by

$$\int_{\mathbb{R}} \mathbf{1}_{w>0}\mathbf{1}_{z>1}ze^{-wz}dz = \mathbf{1}_{w>0}\left[-e^{-wz}\right]_{z=1}^{+\infty} = \mathbf{1}_{w>0}e^{-w}.$$

Hence $W \sim \mathcal{E}(1)$ is the exponential distribution.

The density of Z is given by

$$\begin{aligned}\int_{\mathbb{R}} \mathbf{1}_{w>0}\mathbf{1}_{z>1}we^{-wz}dw &= \mathbf{1}_{z>1}\int_0^{+\infty} we^{-wz}dw \\ &= \mathbf{1}_{z>1}\left[w \times \frac{-1}{z}e^{-wz}\right]_{w=0}^{+\infty} + \frac{\mathbf{1}_{z>1}}{z}\int_0^{+\infty} e^{-wz}dw \\ &= 0 + \frac{\mathbf{1}_{z>1}}{z}\left[\frac{-1}{z}e^{-wz}\right]_{w=0}^{+\infty} = \frac{\mathbf{1}_{z>1}}{z^2}.\end{aligned}$$

Hence Z satisfies the Pareto distribution with parameter 2.

Alternative Solution (by Mengchen Xu). Let $F_W(w) = \text{Prob}(W \le w)$ be the probability distribution of W. Then it is supported on $x > 1 \iff w > 0$, and $\text{Prob}(W \le w) = \text{Prob}(\log X \le \log x) = \text{Prob}(X \le x)$, hence

$$F_W(w) = F_X(x) = \int_1^x \frac{dt}{t^2} = 1 - \frac{1}{x} = 1 - e^{-w}$$

Hence $W \sim \mathcal{E}(1)$ is the exponential distribution with density $\mathbf{1}_{w>0}e^{-w}$.

Now $X, Y > 1$ implies $F_Z(z) = \text{Prob}(Z \le z)$ is supported on $z > 1$. We have

$$\begin{aligned} F_Z(z) &= \text{Prob}(1 + \frac{\log Y}{\log X} \le z) \\ &= \text{Prob}(Y \le X^{z-1}) \\ &= \int_1^{\infty}\int_1^{x^{z-1}} \frac{dy}{y^2}\frac{dx}{x^2} \\ &= \int_1^{\infty} \frac{1}{x^2}\left(1 - \frac{1}{x^{z-1}}\right)dx \\ &= 1 - \frac{1}{z}. \end{aligned}$$

Hence the density is $\frac{\mathbf{1}_{z>1}}{z^2}$ and Z satisfies the Pareto distribution with parameter 2.

Problem 6. Substituting $\mathrm{Li}_2'(x) = -\dfrac{\log(1-x)}{x}$, and integration by parts, we have

$$\begin{aligned}
\int_0^1 \frac{\log(x)\log^2(1-x)}{x}dx &= -\int_0^1 \log x \log(1-x)\mathrm{Li}_2'(x)dx \\
&= -\big[\log x \log(1-x)\mathrm{Li}_2(x)\big]_0^1 + \int_0^1 \left(\frac{\log(1-x)}{x} - \frac{\log x}{1-x}\right)\mathrm{Li}_2(x)dx \\
&= -\int_0^1 \mathrm{Li}_2'(x)\mathrm{Li}_2(x)dx - \int_0^1 \frac{\log x}{1-x}\mathrm{Li}_2(x)dx \\
&= -\frac{1}{2}\mathrm{Li}_2^2(1) - \int_0^1 \sum_{k=0}^{\infty} x^k \log(x) \sum_{n=1}^{\infty} \frac{x^n}{n^2}dx \\
&= -\frac{1}{2}\zeta(2)^2 - \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=0}^{\infty}\int_0^1 x^{n+k}\log(x)dx \\
&= -\frac{1}{2}\zeta(2)^2 + \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=0}^{\infty}\frac{1}{(n+k+1)^2} \\
&= -\frac{1}{2}\zeta(2)^2 + \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=n+1}^{\infty}\frac{1}{k^2}
\end{aligned}$$

where the interchange of integration and summation is justified by Tonelli's Theorem (all the terms are measurable functions and have the same sign).

By symmetry, we can evaluate

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=n+1}^{\infty}\frac{1}{k^2} = \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{n=1}^{\infty}\frac{1}{n^4}\right) = \frac{1}{2}\zeta(2)^2 - \frac{1}{2}\zeta(4)$$

hence the final answer is $-\dfrac{1}{2}\zeta(4) = -\dfrac{\pi^4}{180}$.

Alternative Solution. Writing $\log(1-x) = -\sum_{n=1}^{\infty}\dfrac{x^n}{n}$, we have

$$\begin{aligned}
\int_0^1 \frac{\log x \log^2(1-x)}{x}dx &= \int_0^1 \sum_{n,m=1}^{\infty}\frac{x^n x^m}{nm}\frac{\log x}{x}dx \\
&= -\sum_{n,m=1}^{\infty}\frac{1}{nm}\int_0^1 x^{n+m-1}\log x dx \\
&= -\sum_{n,m=1}^{\infty}\frac{1}{nm(n+m)^2}
\end{aligned}$$

where the interchange of integration and summation is justified by Tonelli's Theorem.

By partial fraction,

$$\begin{aligned}
\sum_{n,m=1}^{\infty}\frac{1}{nm(n+m)^2} &= \sum_{n,m=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+m}\right)\frac{1}{m^2(n+m)} \\
&= \sum_{n,m=1}^{\infty}\frac{1}{m^2 n(n+m)} - \sum_{n,m=1}^{\infty}\frac{1}{m^2}\frac{1}{(n+m)^2}
\end{aligned}$$

The first summation by symmetry gives

$$\sum_{n,m=1}^{\infty} \frac{1}{m^2 n(n+m)} = \frac{1}{2}\left(\sum_{n,m=1}^{\infty} \frac{1}{m^2 n(n+m)} + \frac{1}{n^2 m(n+m)}\right) = \frac{1}{2}\sum_{m,n=1}^{\infty} \frac{1}{m^2 n^2} = \frac{1}{2}\zeta(2)^2$$

while the second summation is the same as previous solution:

$$\sum_{n,m=1}^{\infty} \frac{1}{m^2}\frac{1}{(n+m)^2} = \frac{1}{2}\zeta(2)^2 - \frac{1}{2}\zeta(4).$$

Combining we obtain the answer $-\frac{1}{2}\zeta(4) = -\frac{\pi^4}{180}$.

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