In the following, $\mathbb{R}$ denotes real numbers and $n, k \in \mathbb{N}$ denote positive integers.

Problem 1. Happy N^{th} Birthday to Professor Frederick Fong, where

$$N = \frac{1}{2\pi} \int_0^4 \sqrt{4x^9 - x^{10}}\, dx.$$

Compute N.

Problem 2. Let $\mathbf{A}$ be a real 45×45 matrix. Suppose for every $k \in \mathbb{N}$ there exists a real symmetric matrix $\mathbf{B}_k$ such that

$$2025\mathbf{B}_k = \mathbf{A}^k + \mathbf{B}_k^2.$$

Find the largest possible value of det $\mathbf{A}$.

Problem 3. For any $n \in \mathbb{N}$ and $k \leq n$, let $P_k = \left(\frac{k}{n}, 1 - \frac{k}{n}, 0\right) \in \mathbb{R}^3$ and Q_k be the unique point on the positive z-axis such that $P_k Q_k$ has length 1.

Compute the limit

$$L = \lim_{n\to\infty} \sum_{k=1}^{n-1} V_k$$

where V_k is the volume of the tetrahedron $OP_kP_{k+1}Q_k$.

Problem 4. Fix $n, k \in \mathbb{N}$. Choose a random k-element subset $X \subset \{1, 2, \ldots, k + 2025\}$ and a random n-element subset $Y \subset \{1, 2, \ldots, k + n + 2025\}$ uniformly.

Find the probability

$$\mathbb{P}\left(\min(Y) > \max(X)\right).$$

Problem 5. Consider the following two sequences

$$a_n := \sum_{k=0}^{\infty} \frac{k^n}{k!} \quad \text{and} \quad b_n := \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Prove that $a_n b_n$ is an integer for any $n \in \mathbb{N}$.

Problem 6. Let $\Delta(n) := n^n$. The MEGA number is defined to be

$$\text{MEGA} = \underbrace{\Delta(\Delta(\Delta(\cdots\Delta}_{256 \text{ times}}(256)...).$$

Find the positive integer N such that

$$2 \uparrow\uparrow N < \text{MEGA} < 2 \uparrow\uparrow (N+1)$$

where $2 \uparrow\uparrow n := \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n}$ is the Knuth's up-arrow notation. (E.g. $2 \uparrow\uparrow 4 = 2^{2^{2^2}} = 65536$.)

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