10th HKUST Undergraduate Math Competition – Junior Level

Suggested Solutions

Problem 1.

$\boxed{N = 42.}$

First rearrange as

$$\frac{1}{2\pi}\int_0^4 x^{9/2}\sqrt{4-x}dx.$$

We do the substitution $x = 4\sin^2\theta$ to obtain

$$\frac{2^{12}}{\pi}\int_0^{\frac{\pi}{2}} \sin^{10}\theta\cos^2\theta d\theta = \frac{2^{12}}{\pi}\left(\int_0^{\frac{\pi}{2}} \sin^{10}\theta d\theta - \int_0^{\frac{\pi}{2}} \sin^{12}\theta d\theta\right).$$

This can be evaluated by the formula

$$\int_0^{\frac{\pi}{2}} \sin^{2n}\theta d\theta = \frac{\pi}{2}\frac{(2n-1)!!}{(2n)!!}$$

(which follows e.g. from the reduction formula) to get

$$N = \frac{2^{12}}{\pi}\left(\frac{\pi}{2}\frac{9\cdot 7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2} - \frac{\pi}{2}\frac{11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2}\right) = 42.$$

Problem 2.

$\boxed{\det\mathbf{A} = 1.}$

For $k = 1$, we obtain $\mathbf{A} = 2025\mathbf{B}_1 - \mathbf{B}_1^2$. Since $\mathbf{B}_1$ is real and symmetric, so is $\mathbf{A}$. Thus $\mathbf{A}$ is diagonalizable and all eigenvalues of $\mathbf{A}$ are real.

Now fix a positive integer k and let λ be an real eigenvalue of $\mathbf{A}$. Considering the diagonal form of both $\mathbf{A}$ and $\mathbf{B}_k$, we know that there exists a real eigenvalue μ of $\mathbf{B}_k$ such that

$$2025\mu = \lambda^k + \mu^2 \Longrightarrow \mu^2 - 2025\mu + \lambda^k = 0.$$

The last equation is a second degree equation with a real root. Therefore, the discriminant is non-negative:

$$2025^2 - 4\lambda^k \geq 0 \Longrightarrow \lambda^k \leq \frac{2025^2}{4}.$$

If $|\lambda| > 1$, letting k even and sufficiently large we reach a contradiction. Thus $|\lambda| \leq 1$.

Finally, since $\det A$ is the product of the eigenvalues of A and each of them has absolute value less than or equal to 1, we get $|\det\mathbf{A}| \leq 1$.

The maximal possible value of $\det(\mathbf{A}) = 1$ can be reached by considering the identity matrix $\mathbf{A} = \mathbf{I}$ with $\mathbf{B}_k$ a diagonal matrix with diagonal entries the real roots of $\lambda^2 - 2025\lambda + 1 = 0$.

Remark. The size of $\mathbf{A}$ is irrelevant.

Problem 3.

$$\boxed{L = \frac{\sqrt{2}}{48}\pi.}$$

Let S_k be the area of the triangle OP_kP_{k+1}. It is easy to compute

$$S_k = \frac{1}{2}\left|\frac{k}{n}\left(1-\frac{k+1}{n}\right) - \frac{k+1}{n}\left(1-\frac{k}{n}\right)\right| = \frac{1}{2n}.$$

Furthermore,

$$OQ_k = \sqrt{P_kQ_k^2 - OP_k^2} = \sqrt{1-\left(\left(\frac{k}{n}\right)^2 + \left(1-\frac{k}{n}\right)^2\right)} = \sqrt{2\left(\frac{k}{n} - \left(\frac{k}{n}\right)^2\right)}$$

so that

$$V_k = \frac{1}{3}S_k \cdot OQ_k = \frac{1}{6n}\sqrt{2\left(\frac{k}{n} - \left(\frac{k}{n}\right)^2\right)}$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{n-1} V_k = \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{6n}\sqrt{2\left(\frac{k}{n} - \left(\frac{k}{n}\right)^2\right)} = \frac{\sqrt{2}}{6}\int_0^1 \sqrt{x-x^2}dx$$

as a Riemann sum.

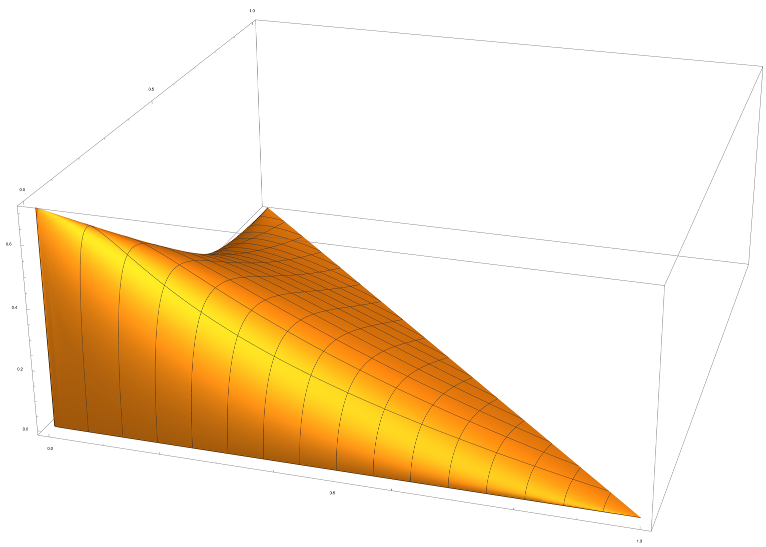
Note that

$$\int_0^1 \sqrt{x-x^2}dx = \frac{\pi}{8}$$

being the area of the semicircle with radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$, so the answer to the problem is

$$\lim_{n\to\infty}\sum_{k=1}^{n-1} V_k = \frac{\sqrt{2}}{48}\pi.$$

Remark. Under the limit, the sum approximates the volume of the solid under the graph $f(x,y) = \dfrac{\sqrt{2xy}}{x+y}(1-x-y)$.



Problem 4.

$$\boxed{\mathbb{P} = \frac{1}{\binom{n+k}{k}}}.$$

The number of choices for (X, Y) is $\binom{k+2025}{k} \cdot \binom{n+k+2025}{n}$.

The number of such choices with $\min(Y) > \max(X)$ is equal to $\binom{n+k+2025}{n+k}$ since this is the number of choices for the $(n+k)$-element set $Z = X \sqcup Y$ and this union together with the condition $\min(Y) > \max(X)$ determines X and Y uniquely (note in particular that no elements of X will be larger than $k + 2025$). Hence the probability is

$$\mathbb{P} = \frac{\binom{n+k+2025}{n+k}}{\binom{k+2025}{k} \cdot \binom{n+k+2025}{n}} = \frac{1}{\binom{n+k}{k}}$$

where the identity can be seen by expanding the binomial coefficients on both sides into factorials and canceling.

Problem 5. From the Taylor series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

we conclude that

$$a_n = \left(x\frac{d}{dx} \right)^n (e^x) \bigg|_{x=1}, \qquad b_n = \left(x\frac{d}{dx} \right)^n (e^{-x}) \bigg|_{x=1}.$$

By induction we easily see that

$$\left(x\frac{d}{dx} \right)^n (e^x) = p_n(x)e^x, \qquad \left(x\frac{d}{dx} \right)^n (e^{-x}) = q_n(x)e^{-x}$$

for some polynomials $p_n(x), q_n(x)$ with integer coefficients.

In particular a_n and b_n are given by

$$a_n = p_n(1)e, \qquad b_n = q_n(1)e^{-1}$$

hence $a_n b_n = p_n(1)q_n(1)$ is an integer.

Problem 6.

$\boxed{N = 259.}$

We claim by induction that for $n \geq 3$,

$$2 \uparrow\uparrow n < \underbrace{\Delta \cdots \Delta}_{n-3}(256) \leq \sqrt{2 \uparrow\uparrow (n+1)} < 2 \uparrow\uparrow (n+1).$$

The base case $n = 3$ is trivial: We have

$$2 \uparrow\uparrow 3 = 16 < 256 = \sqrt{2 \uparrow\uparrow 4} < 2 \uparrow\uparrow 4.$$

Note that Δ is increasing. Apply to the inequality, we compute

$$\Delta(2 \uparrow\uparrow n) = (2 \uparrow\uparrow n)^{2\uparrow\uparrow n} = 2^{2\uparrow\uparrow(n-1)\cdot 2\uparrow\uparrow n} = 2^{2^{2\uparrow\uparrow(n-2)+2\uparrow\uparrow(n-1)}} > 2^{2^{2\uparrow\uparrow(n-1)}} = 2 \uparrow\uparrow (n+1)$$

so the left inequality is satisfied.

On the other hand,

$$\begin{aligned} \Delta(\sqrt{2 \uparrow\uparrow (n+1)}) = \sqrt{2 \uparrow\uparrow (n+1)}^{\sqrt{2\uparrow\uparrow(n+1)}} &\leq \sqrt{2 \uparrow\uparrow (n+2)} \\ \log_2 \iff 2^{2^{2\uparrow\uparrow(n-1)-1}} \cdot 2^{2\uparrow\uparrow(n-1)-1} &\leq 2^{(2\uparrow\uparrow n)-1} \\ \log_2 \iff 2^{2\uparrow\uparrow(n-1)-1} + 2 \uparrow\uparrow (n-1) &\leq 2 \uparrow\uparrow n \end{aligned}$$

which is true since for $n \geq 3$,

$$2^{2\uparrow\uparrow(n-1)-1} + 2 \uparrow\uparrow (n-1) = 2^{2\uparrow\uparrow(n-1)-1} + 2^{2\uparrow\uparrow(n-2)} \leq 2^{2\uparrow\uparrow(n-1)-1} + 2^{2\uparrow\uparrow(n-1)-1} = 2 \uparrow\uparrow n.$$

So the right inequality holds as well and we finish the induction.

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