In the following, $\mathbb{R}$ denotes the real numbers and $n, m \in \mathbb{N}$ denote positive integers.

Problem 1. Devan celebrates his birthday today by drawing two real numbers $x, y \in (0,1)$ at random uniformly. It will bring him good luck if he can guess correctly whether the closest integer to $\dfrac{x}{y}$ is even or odd.

Which parity (even or odd) should Devan pick for a better chance of being lucky today, and what is the corresponding probability?

Problem 2. Let $p, q \in \mathbb{N}$ be two prime numbers. Determine explicitly the group presented by

$$G := \langle x, y | x^p = y^q = xyxyx = 1 \rangle$$

where 1 denotes the identity element.

Problem 3. Let $p(x) \in \mathbb{R}[x]$ be a polynomial of degree n having only real zeros.

Prove that for any $x \in \mathbb{R}$ the following inequality holds:

$$(n-1)(p'(x))^2 \geq np(x)p''(x).$$

For which $p(x)$ does equality hold?

Problem 4. Let $a, b > 0$ be two real numbers. Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{e^{ax} + e^{-bx}}.$$

Problem 5. Prove that there exists an infinite number of relatively prime pairs (m, n) of positive integers such that the equation

$$(x+m)^3 = nx$$

has three distinct integer roots.

Problem 6. Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{Z}$. Suppose a function $f : \mathbb{Z} \to \mathbb{R}$ satisfies

$$\sum_{i=1}^{n} f(k + a_i \ell) = 0$$

for any integers $k, \ell \in \mathbb{Z}$ with $\ell \neq 0$. Prove that f is identically zero.

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