10th HKUST Undergraduate Math Competition – Senior Level

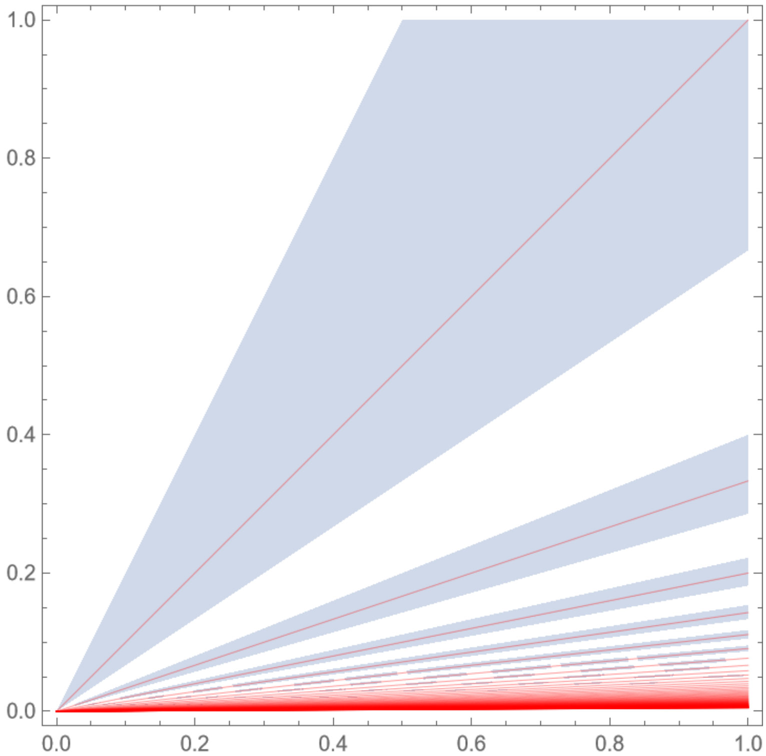
Suggested Solutions

Problem 1.

$$\boxed{\text{Odd with probability } P = \frac{\pi - 1}{4}.}$$

It is equivalent to compute the shaded (odd) / unshaded (even) area within the unit square separated by the boundary

$$\frac{x}{y} = 2n + \frac{1}{2}, \qquad n = 0, 1, 2, ...$$



It is easy to see that the shaded area is larger than the unshaded area, since each subsquent triangle has a shorter base. So $\frac{x}{y}$ has a higher chance of being odd.

To compute the actual probability, the largest shaded region is split into two triangles of height 1 and width $\frac{1}{2}$ and $\frac{1}{3}$ respectively, giving the area $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

The rest of the shaded area are triangles of height 1 and width $\frac{1}{2n+\frac{1}{2}} - \frac{1}{2n+\frac{3}{2}} = \frac{2}{4n+1} - \frac{2}{4n+3}$ for $n = 1, 2, ...$ giving a total of

$$P(odd) = \frac{5}{12} + \left(\frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right) = \frac{5}{12} + \left(\frac{\pi}{4} - \frac{2}{3}\right) = \frac{\pi - 1}{4}$$

Problem 2.

Let $z = xy$. Then note that

$$x^{-1} = xyxy = z^2 \Longrightarrow x = z^{-2}$$

$$xyxyxy = y \Longrightarrow y = z^3$$

In particular $\{z\}$ is a generator of G, e.g. $G = \langle z \rangle$ is cyclic, where $z^{2p} = z^{3q} = 1$.

So G is cyclic of order $gcd(2p, 3q) = \begin{cases} 6 & (p,q) = (3,2) \\ p & p = q \\ 3 & p = 3, q \neq 2 \\ 2 & p \neq 3, q = 2 \\ 1 & otherwise \end{cases}$.

Problem 3.

If $n = 1$, then the inequality is trivial as the right hand side is zero.

Now suppose $n > 1$. Let $x_1, \ldots, x_n$ be the zeros of $p(x)$.

Clearly the inequality is true when $x = x_i$ is one of the roots, and equality is possible only if $p'(x_i) = 0$, i.e., if x_i is a multiple zero of $p(x)$.

Now suppose that x is not a zero of $p(x)$. Using the identities

$$\frac{p'(x)}{p(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i}, \quad \frac{p''(x)}{p(x)} \sum_{1 \le i < j \le n} \frac{2}{(x - x_i)(x - x_j)},$$

we find

$$(n-1)\left(\frac{p'(x)}{p(x)}\right)^2 - n\frac{p''(x)}{p(x)} = \sum_{i=1}^{n} \frac{n-1}{(x - x_i)^2} - \sum_{1 \le i < j \le n} \frac{2}{(x - x_i)(x - x_j)}.$$

The last expression is simply

$$\sum_{1 \le i < j \le n} \left(\frac{1}{x - x_i} - \frac{1}{x - x_j}\right)^2 \ge 0.$$

So the inequality is proved.

From the last line, we see that for equality to hold for every real x, it is necessary that $x_1 = x_2 = \ldots = x_n$, that is, $p(x) = c(x - x_1)^n$ for some real constant $c \in \mathbb{R}$.

Problem 4. $\boxed{\dfrac{\pi}{a+b}\dfrac{1}{\sin\frac{b\pi}{a+b}} \text{ or } \dfrac{\pi}{a+b}\dfrac{1}{\sin\frac{a\pi}{a+b}}.}$

By Complex Analysis. Rewrite it as

$$I = \int_{-\infty}^{\infty} \frac{e^{bx}dx}{1+e^{(a+b)x}}$$

We consider the contour integral over the rectangle $[-R, R] \times [0, \frac{2\pi i}{a+b}] \subset \mathbb{C}$ going counterclockwise.

$$\left(\int_{-R}^{R} + \int_{R}^{R+\frac{2\pi i}{a+b}} + \int_{R+\frac{2\pi i}{a+b}}^{-R+\frac{2\pi i}{a+b}} + \int_{-R+\frac{2\pi i}{a+b}}^{-R}\right) \frac{e^{bz}}{1+e^{(a+b)z}}dz$$

Note by change of variable $z \mapsto z + \frac{2\pi i}{a+b}$, we have

$$\int_{R+\frac{2\pi i}{a+b}}^{-R+\frac{2\pi i}{a+b}} \frac{e^{bz}}{1+e^{(a+b)z}}dz = -e^{\frac{2\pi ib}{a+b}} \int_{-R}^{R} \frac{e^{bz}}{1+e^{(a+b)z}}dz$$

Under the limit $R \to \infty$, the integrals $\int_{R}^{R+\frac{2\pi i}{a+b}}$ and $\int_{-R+\frac{2\pi i}{a+b}}^{-R}$ vanishes by exponential decay (since the integrand grows like e^{-az} and e^{-bz} respectively), and by the Residue Theorem we have

$$I - e^{\frac{2\pi ib}{a+b}} I = 2\pi i \cdot \mathrm{Res}$$

The integrand $\frac{e^{bx}dx}{1+e^{(a+b)x}}$ has only one simple pole at $z = \frac{\pi i}{a+b}$ with residue

$$\lim_{z \to \frac{\pi i}{a+b}} (z - \frac{\pi i}{a+b}) \frac{e^{bz}}{1+e^{(a+b)z}} = -\frac{1}{a+b}e^{\frac{\pi ib}{a+b}}$$

Hence rearranging we obtain

$$I = \frac{2\pi i(-\frac{1}{a+b}e^{\frac{\pi ib}{a+b}})}{1-e^{\frac{2\pi ib}{a+b}}} = \frac{\pi}{a+b}\frac{1}{\sin\frac{b\pi}{a+b}}.$$

By Real Analysis. By a substitution $t = e^{(a+b)x}$ the integral becomes

$$\frac{1}{a+b}\int_0^{\infty} \frac{t^{-\frac{a}{a+b}}}{t+1}dt$$

Let $k = -\frac{a}{a+b}$ so that $-1 < k < 0$. By substitution $t = \frac{u}{1-u}$ again, the integral becomes

$$\frac{1}{a+b}\int_0^1 u^k(1-u)^{-k-1}du = \frac{1}{a+b}B(k+1, -k)$$

where the beta integral

$$B(k+1, -k) = \frac{\Gamma(k+1)\Gamma(-k)}{\Gamma(1)} = \Gamma(k+1)\Gamma(-k).$$

Using the reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ and set $z = -k$ we obtain the final answer

$$\frac{1}{a+b} \cdot \frac{\pi}{\sin\frac{\pi a}{a+b}}.$$

Problem 5.

Substituting $y = x + m$, we can replace the equation by

$$y^3 - ny + mn = 0.$$

Let two roots be u and v; the third one must be $w = -(u+v)$ since the sum $u+v+w$ is 0. The roots must also satisfy

$$uv + uw + vw = -(u^2 + uv + v^2) = -n \Longrightarrow u^2 + uv + v^2 = n$$

and

$$uvw = -uv(u+v) = -mn.$$

So we need some integer pairs (u, v) such that $uv(u+v)$ is divisible by $u^2 + uv + v^2$.

We consider the family where $v = qu$ is an integer multiple of u, then we get

$$u^2 + uv + v^2 = u^2(1 + q + q^2),$$

and

$$uv(u+v) = u^3 q(1+q).$$

Hence setting $u = 1 + q + q^2$ we have

$$m = \frac{uv(u+v)}{u^2 + uv + v^2} = q + q^2.$$

Substituting back to the original quantites, we obtain the family of parameters

$$n = (1 + q + q^2)^3, \qquad m = q + q^2,$$

which are clearly coprime, and the three distinct roots of the original equations are

$$x_1 = 1, \qquad x_2 = q^3, \qquad x_3 = -(1+q)^3.$$

Problem 6.

Let $M \geq 0$ be an integer such that $a_i' = a_i + M \geq 0$ for all i. Then for any k, ℓ with $\ell \neq 0$,

$$\sum_{i=1}^{n} f(k + a_i'\ell) = \sum_{i=1}^{n} f((k+M) + a_i\ell) = 0$$

by assumption, hence WLOG we may assume all $a_i \in \mathbb{N}$.

Let us define a subset $\mathcal{I}$ of the polynomial ring $\mathbb{R}[X]$ as follows:

$$\mathcal{I} := \left\{ P(X) = \sum_{j=0}^{m} b_j X^j : \sum_{j=0}^{m} b_j f(k + j\ell) = 0 \quad \text{for all } k, \ell \in \mathbb{Z}, \ell \neq 0 \right\}.$$

This is a subspace of the real vector space $\mathbb{R}[X]$. Furthermore, $P(X) \in \mathcal{I}$ implies $X \cdot P(X) \in \mathcal{I}$. Hence, $\mathcal{I}$ is an ideal, and it is non-zero, because the polynomial $R(X) = \sum_{i=1}^{n} X^{a_i} \in \mathcal{I}$.

Recall that $\mathbb{R}[X]$ is a principal ideal domain. Thus, $\mathcal{I}$ is generated (as an ideal) by some non-zero polynomial Q.

If Q is constant then $1 \in \mathcal{I}$, which by definition of $\mathcal{I}$ implies $f(k) = 0$ for all $k \in \mathbb{Z}$, hence f is identically zero and we are done.

Otherwise we may assume Q has a complex zero $c \in \mathbb{C}$. Again, by the definition of $\mathcal{I}$, the polynomial $Q(X^m)$ belongs to $\mathcal{I}$ for every natural number $m \geq 1$; hence $Q(X)$ divides $Q(X^m)$. This shows that all the complex numbers

$$c, c^2, c^3, c^4, \cdots$$

are roots of Q. Since Q can have only finitely many roots, we must have $c^N = 1$ for some $N \geq 1$; in particular, $Q(1) = 0$, which implies $P(1) = 0$ for all $P \in \mathcal{I}$. This contradicts the fact that $R(X) = \sum_{i=1}^{n} X^{a_i} \in \mathcal{I}$ since $R(1) = n \neq 0$, and we are done.

–End of Paper–