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DEPARTMENT OF MATHEMATICS

11th HKUST Undergraduate Math Competition – Junior Level

April 25th, 2026

Suggested Solutions

Remark. Most of the 2026's in this competition are red herrings.
Throughout the solution, let $N = 2026$.

Problem 1.

$\boxed{k = 10^{2026} + 1}$

First observe that the number $10^N - 1$ is a string of N 9's, so its digit sum is $9N = 18234$.

We show that for all $1 \le k \le 10^N$ the product $k \times (10^N - 1)$ has digit sum $9N$, while for $k = 10^N + 1$ the digit sum changes to $18N = 36468 > 33333$.

- Let $1 \le k \le 10^N$. Write $k - 1$ as an N-digit decimal number (allowing leading zeros)

$$k - 1 = (a_N a_{N-1} \cdots a_1)_{10}, \qquad (a_i \in \{0, ..., 9\})$$

  and let $b_i := 9 - a_i$. Then we get the decimal representation

$$\begin{aligned} k \times (10^N - 1) &= k \cdot 10^N - k = (k-1) \cdot 10^N + (10^N - k) \\ &= (a_N a_{N-1} \cdots a_1 b_N b_{N-1} \cdots b_1)_{10} \end{aligned}$$

  as a $2N$-digit number. Hence the total digit sum is $\sum_{i=1}^{N} (a_i + b_i) = 9N$ as required.

- On the other hand, if $k = 10^N + 1$, then

$$k \times (10^N - 1) = 10^{2N} - 1$$

  is a string of $2N$ 9's, so its digit sum is $18N$ instead.

Problem 2.

By the rank-nullity theorem, we have

$$\operatorname{rank}(\mathbf{A}) + \dim\ker(\mathbf{A}) = N$$

$$\operatorname{rank}(\mathbf{A}-\mathbf{I}) + \dim\ker(\mathbf{A}-\mathbf{I}) = N$$

Therefore we also have

$$\dim\ker(\mathbf{A}) + \dim\ker(\mathbf{A}-\mathbf{I}) = N.$$

On the other hand, if $\mathbf{v} \in \ker(\mathbf{A}) \cap \ker(\mathbf{A}-\mathbf{I})$, then

$$\mathbf{A}\mathbf{v} = \mathbf{A}\mathbf{v} - \mathbf{v} \Longrightarrow \mathbf{v} = \mathbf{0}$$

hence $\ker(\mathbf{A}) \cap \ker(\mathbf{A}-\mathbf{I}) = \{\mathbf{0}\}$.

Therefore we have a direct sum

$$\mathbb{R}^N = \ker(\mathbf{A}) \oplus \ker(\mathbf{A}-\mathbf{I})$$

In particular, for any $\mathbf{v} \in \mathbb{R}^N$, we can uniquely write $\mathbf{v} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a} \in \ker(\mathbf{A})$ and $\mathbf{b} \in \ker(\mathbf{A}-\mathbf{I})$, and we obtain

$$\mathbf{A}^2\mathbf{v} = \mathbf{A}\mathbf{A}(\mathbf{a}+\mathbf{b}) = \mathbf{A}(\mathbf{A}\mathbf{a}+\mathbf{A}\mathbf{b}) = \mathbf{A}(\mathbf{0}+\mathbf{b}) = \mathbf{0}+\mathbf{A}\mathbf{b} = \mathbf{A}(\mathbf{a}+\mathbf{b}) = \mathbf{A}\mathbf{v}.$$

Hence $\mathbf{A}^2 = \mathbf{A}$ is a projection matrix, and $\mathbf{A}^{2026} = \mathbf{A}$ can only have 0 or 1 as eigenvalues.

Problem 3.

$$\boxed{P = \left(\frac{2027}{2^{2026}}\right)^{2026}}$$

WLOG, write $S = \{1, 2, ..., N\}$.

Define the $N \times N$ square matrix $\mathbf{A} = (a_{ij})$ where $a_{ij} = \begin{cases} 1 & i \in A_j \\ 0 & i \notin A_j \end{cases}$.

Then the condition becomes the probability of finding such matrix of 0's and 1's where each row forms a non-decreasing sequence of 0's and 1's.

For each row, the total possible combination is 2^N, while we observe that there are exactly $N + 1$ nondecreasing sequences of 0's and 1's (from $(0 \cdots 000), (0 \cdots 001), (0 \cdots 011)$ all the way up to $(1 \cdots 111)$), and all N rows are independent, hence the probability is $\left(\frac{N+1}{2^N}\right)^N$.

Problem 4.

$$\boxed{a_{2026} = \frac{11!}{15} = 2661120}$$

Observe that the powers k_i are all possible ways of expressing a number in base 3 consisting of 0 and 1 only, and also with 0 at the unit position. (Since we do not have x^{3^0} power.)

The first few powers thus obtained are

$$k_1 = 10_3 = 3, \qquad k_2 = 100_3 = 9, \qquad k_3 = 110_3 = 12, \qquad k_4 = 1000_3 = 27,$$

$$k_5 = 1010_3 = 30, \qquad k_6 = 1100_3 = 36, \qquad k_7 = 1110_3 = 39, \qquad k_8 = 10000_3 = 81, \qquad \textit{etc.}$$

In other words, ordering the powers of k_i is equivalent to writing i in base 2, attaching a 0 at the end, and reading the result as a number in base 3.

On the other hand, a_i is the product of the exponents n of the powers 3^n used in k_i.

In particular

$$2026 = 11111101010_2 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3 + 2^1$$

Hence we have

$$k_{2026} = 111111010100_3 = 3^{11} + 3^{10} + 3^9 + 3^8 + 3^7 + 3^6 + 3^4 + 3^2$$

and $a_{2026} = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 4 \cdot 2 = \dfrac{11!}{15} = 2661120.$

Problem 5.

$$\boxed{-\sqrt{5+2\sqrt{5}}}$$

Do a change of variable $x = \tan u$, we find that

$$f(x) = \frac{3\tan u - \tan^3 u}{1 - 3\tan^2 u} = \tan 3u.$$

Therefore

$$f(\tan u) = \tan 3u$$

and iteratively

$$f(f(x)) = f(\tan 3u) = \tan 9u, \qquad ..., \qquad f^n(x) = \tan 3^n u = \tan(3^n \arctan x).$$

We recall that $\arctan\left(\sqrt{5+2\sqrt{5}}\right) = \frac{2\pi}{5}$ and we easily observe that $3^{2026} \equiv 4 \pmod 5$, hence

$$f^{2026}\left(\sqrt{5+2\sqrt{5}}\right) = \tan\left(3^{2026}\frac{2\pi}{5}\right) = \tan\frac{8\pi}{5} = -\sqrt{5+2\sqrt{5}}.$$

Alternatively. One can compute by brute force (very carefully) that

$$\begin{aligned}
f\left(\sqrt{5+2\sqrt{5}}\right) &= \sqrt{5-2\sqrt{5}} \\
f\left(\sqrt{5-2\sqrt{5}}\right) &= -\sqrt{5+2\sqrt{5}} \\
f\left(-\sqrt{5+2\sqrt{5}}\right) &= -\sqrt{5-2\sqrt{5}} \\
f\left(-\sqrt{5-2\sqrt{5}}\right) &= \sqrt{5+2\sqrt{5}}
\end{aligned}$$

and hence $\sqrt{5+2\sqrt{5}}$ is a fixed point of f^4, and we can use the fact that $2026 \equiv 2 \pmod 4$ to obtain

$$f^{2026}\left(\sqrt{5+2\sqrt{5}}\right) = f^2\left(\sqrt{5+2\sqrt{5}}\right) = -\sqrt{5+2\sqrt{5}}.$$

Problem 6.

2026

Only the leading terms of the polynomials matter:

$$f(x) = \frac{1}{N}x^N + \cdots, \qquad \text{and} \qquad h(x) := \frac{g(x)}{x} = Nx^{N-1} + \cdots$$

Observe that

$$\lim_{x \to +\infty} \int_0^x e^{f(t)} dt = +\infty$$

and

$$\lim_{x \to +\infty} \frac{e^{f(x)}}{h(x)} = +\infty.$$

Therefore we apply l'Hôpital's rule to the expression

$$\begin{aligned} \lim_{x \to +\infty} \frac{\int_0^x e^{f(t)} dt}{\frac{e^{f(x)}}{h(x)}} &= \lim_{x \to +\infty} \frac{\frac{d}{dx} \int_0^x e^{f(t)} dt}{\frac{d}{dx} \frac{e^{f(x)}}{h(x)}} \\ &= \lim_{x \to +\infty} \frac{h(x)^2}{f'(x)h(x) - h'(x)} \\ &= \lim_{x \to +\infty} \frac{N^2 x^{2N} + \cdots}{N x^{2N} + \cdots} \\ &= N \end{aligned}$$

exists, hence the required limits hold.

–End of Paper–