11th HKUST Undergraduate Math Competition – Senior Level

April 25th, 2026

INSTRUCTIONS

- Do **NOT** turn over the page until instructed to do so.
- This is a **CLOSED BOOK, CLOSED NOTES** competition.
- Calculators are **NOT** allowed.
- Provide complete details of your solution. You may use results from **any UG courses** with proper citations.
- Each problem carries equal weight.

Time allowed: 3 hours.

Problem 1. Let $p > 2026$ be a prime number, and let $m, n \in \mathbb{N}$ be such that

$$\frac{m}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$$

expressed in lowest term. Prove that m is divisible by p^2.

Problem 2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence satisfying

$$a_{n+m} \leq a_n + a_m + \frac{2026}{mn}$$

for all $m, n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} \frac{a_n}{n}$ exists or diverges to $-\infty$.

Problem 3. Let $\gamma : \mathbb{R} \to S^2 \subset \mathbb{R}^3$ be a smooth curve on the unit sphere with $|\gamma'(s)| = 1$ such that

$$\langle \gamma''(s), \gamma(s) \times \gamma'(s) \rangle = 2026.$$

Show that $\gamma(s)$ lies on a circle on S^2.

(Here $\langle \mathbf{u}, \mathbf{v} \rangle$ and $\mathbf{u} \times \mathbf{v}$ denote the usual inner and cross product respectively in $\mathbb{R}^3$.)

Problem 4. Let $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ be the sequence defined recursively by $a_0 := 1$ and

$$a_{n+1} := \frac{1}{n+1} \sum_{k=0}^{n} \frac{a_k}{n-k+2}, \qquad n \geq 0.$$

Evaluate the infinite series

$$\sum_{k=0}^{\infty} \frac{a_k}{2026^k}.$$

Problem 5. Let $R = \mathbb{Z}/42\mathbb{Z}$ and $f(x) = x$ in $R[x]$. Show that $f(x)$ is reducible in $R[x]$, and determine all its factorizations into irreducibles (up to units).

(Recall that an element $f \in R[x]$ is called **irreducible** iff $f = ab$ implies a or b is a unit (i.e. invertible) in $R[x]$.)

Problem 6. Let f be a continuous function on $\mathbb{R}$ and let $J_1, J_2, J_3, ..., J_{2026} \subset \mathbb{R}$ be closed intervals. Assume $J_{i+1} \subseteq f(J_i)$ for $i = 1, ..., 2025$ and $J_1 \subseteq f(J_{2026})$.

Show that the composite f^{2026} has a fixed point in J_1, i.e. there exists $x_* \in J_1$ such that

$$f^{2026}(x_*) := \underbrace{(f \circ f \circ \cdots \circ f)}_{2026}(x_*) = x_*.$$

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