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DEPARTMENT OF MATHEMATICS

11th HKUST Undergraduate Math Competition – Senior Level

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Suggested Solutions

Problem 1.

Remark. One may recognize this result as the **Wolstenholme's congruence** from elementary number theory.

Since p is an odd prime, we can pair the summands $\frac{1}{k}$ and $\frac{1}{p-k}$ for $k = 1, 2, \ldots, p-1$:

$$\frac{1}{k} + \frac{1}{p-k} = \frac{p}{k(p-k)}.$$

Hence summing over all k gives

$$\sum_{k=1}^{p-1} \frac{1}{k} = \frac{p}{2} \sum_{k=1}^{p-1} \frac{1}{k(p-k)}.$$

Hence we have an equation as integers

$$2m = p \sum_{k=1}^{p-1} \frac{n}{k(p-k)}.$$

Since p is odd, it suffices to show that p divides $S = \sum_{k=1}^{p-1} \frac{n}{k(p-k)}$, which is an integer since the denominator is not divisible by p.

We observe that $p - k \equiv -k \pmod{p}$ and also $\{1^{-1}, ..., (p-1)^{-1}\} \equiv \{1, ..., p-1\} \pmod{p}$ is the same set of residues. Hence

$$S \equiv -\sum_{k=1}^{p-1} \frac{n}{k^2} \equiv -n \sum_{r=1}^{p-1} r^2 = -n \frac{(p-1)p(2p-1)}{6} \equiv 0 \pmod{p}$$

since $p > 3$ so that p does not divide 6.

Problem 2.

Note that we have $a_{n+m} \le a_n + a_m + 2026$ and this is actually the version we need.

Define $b_n := a_n - 2026$. Then it is equivalent to solving for $\lim_{n\to\infty} \frac{b_n}{n}$ where

$$b_{n+m} \le b_n + b_m$$

i.e. subadditive. It suffices to show that

$$\limsup_{n\to\infty} \frac{b_n}{n} \le \liminf_{n\to\infty} \frac{b_n}{n} < \infty.$$

Remark. One may recognize this as the **Fekete's Lemma.**

It is clear by induction, $b_n \le nb_1$, hence $\frac{b_n}{n} \le b_1$ is bounded and $I = \liminf_{n\to\infty} \frac{b_n}{n} < \infty$ exists or equals $-\infty$.

On the other hand, fix any $k \in \mathbb{N}$, write $n = qk + r$ with $1 \le r \le k$. Then

$$b_n \le qb_k + b_r$$

so that

$$\frac{b_n}{n} \le \frac{q}{qk+r} b_k + \frac{b_r}{n} \le \frac{b_k}{k} + \frac{b_r}{n}$$

Taking $n \to \infty$, since $\{b_r\}$ is a finite set, hence bounded, we obtain

$$\limsup_{n\to\infty} \frac{b_n}{n} \le \frac{b_k}{k}.$$

Since this is true for any $k \in \mathbb{N}$, we conclude that

$$\limsup_{n\to\infty} \frac{b_n}{n} \le \inf_k \frac{b_k}{k} \le \liminf_{k\to\infty} \frac{b_k}{k} < \infty$$

as required.

Problem 3. (We omit the variable s for simplicity.) Let $\kappa = 2026$.

Idea of thoughts. One can observe that, if $\gamma(s)$ is a curve on a circle on S^2, then (up to rotation) it is of the form

$$\gamma(s) = \langle c\cos\frac{s}{c}, c\sin\frac{s}{c}, d\rangle, \qquad \text{where } c = \tfrac{1}{\sqrt{1+\kappa^2}}, d = \tfrac{\kappa}{\sqrt{1+\kappa^2}}.$$

In particular it lies on the plane $\frac{1}{c}z = \kappa$.

To prove the converse, the idea is to express the upward vector $V = \langle 0, 0, \frac{1}{c}\rangle$ in terms of $\gamma(s)$.

Observe that since $\langle\gamma,\gamma\rangle = 1$ is a constant, $\langle\gamma,\gamma'\rangle = 0$.

Similarly $\langle\gamma',\gamma'\rangle = 1$ implies $\langle\gamma',\gamma''\rangle = 0$.

In particular, $\{\gamma,\gamma',\gamma\times\gamma'\}$ forms an orthonormal basis of $\mathbb{R}^3$, and by assumption we have

$$\gamma'' = \langle\gamma'',\gamma\rangle\gamma + \kappa(\gamma\times\gamma').$$

Let

$$V(s) := \gamma(s)\times\gamma'(s) + \kappa\gamma(s)$$

we claim that $V(s)$ is a constant. Differentiating s gives

$$\begin{aligned} V'(s) &= \gamma'\times\gamma' + \gamma\times\gamma'' + \kappa\gamma' \\ &= \gamma\times\big(\langle\gamma'',\gamma\rangle\gamma + \kappa(\gamma\times\gamma')\big) + \kappa\gamma' \\ &= \kappa\gamma\times(\gamma\times\gamma') + \kappa\gamma' \end{aligned}$$

Observe that since $\gamma\perp\gamma'$, we have $\gamma\times(\gamma\times\gamma') = -\gamma'$ hence

$$V'(s) = \kappa(-\gamma') + \kappa\gamma' = 0$$

and $V(s) = V$ is a constant vector in $\mathbb{R}^3$.

Finally for all $s\in\mathbb{R}$,

$$\langle\gamma(s), V\rangle = \langle\gamma,\gamma\times\gamma'\rangle + \kappa|\gamma|^2 = \kappa.$$

This shows that the image of γ lies in the plane

$$\{x\in\mathbb{R}^3 \;:\; \langle x, V\rangle = \kappa\}$$

Hence γ is contained in the intersection of this plane with the unit sphere, which is a circle.

Problem 4.

$$\boxed{e \cdot \left(\frac{2025}{2026}\right)^{2025}}$$

Consider the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$. By induction, $0 < a_n \leq 1$, hence the series is absolutely convergent for $|x| < 1$ with $f(0) = 1$ and the function is positive in the interval $[0, 1)$. We need to compute $f(\frac{1}{2026})$.

By the recurrence formula, for $0 < x < 1$,

$$f'(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a_k}{n-k+2}x^n = \sum_{k=0}^{\infty} a_k x^k \sum_{n=k}^{\infty} \frac{x^{n-k}}{n-k+2} = f(x)\sum_{m=0}^{\infty} \frac{x^m}{m+2}.$$

where we can interchange the summation order since the double sum is absolutely convergent. Also recall that we can integrate power series term-by-term.

Hence

$$\begin{aligned}
\log f(x) = \log f(x) - \log f(0) = \int_0^x \frac{f'(t)}{f(t)} dt &= \sum_{m=0}^{\infty} \frac{x^{m+1}}{(m+1)(m+2)} \\
&= \sum_{m=0}^{\infty} \left(\frac{x^{m+1}}{m+1} - \frac{x^{m+1}}{m+2} \right) \\
&= 1 + \left(1 - \frac{1}{x}\right) \sum_{m=0}^{\infty} \frac{x^{m+1}}{m+1} \\
&= 1 + \left(1 - \frac{1}{x}\right) \log \frac{1}{1-x}.
\end{aligned}$$

Therefore $\log f\left(\frac{1}{2026}\right) = 1 - 2025 \log \frac{2026}{2025}$, and our answer is

$$f\left(\frac{1}{2}\right) = e \cdot \left(\frac{2025}{2026}\right)^{2025}.$$

Problem 5.

$f(x) = (21x + 22)(28x + 15)(36x + 7)$ up to units.

Since $2, 3, 7$ are pairwise coprime, by Chinese Remainder Theorem

$$\psi : (\mathbb{Z}/42\mathbb{Z})[x] \simeq (\mathbb{Z}/2\mathbb{Z})[x] \times (\mathbb{Z}/3\mathbb{Z})[x] \times (\mathbb{Z}/7\mathbb{Z})[x]$$

where the ring structure on RHS is componentwise, and a polynomial in LHS is irreducible if and only if it is irreducible on RHS, which means it is irreducible in one component and units in the other two.

Note that $\overline{f}(x) = x$ is irreducible in $(\mathbb{Z}/p\mathbb{Z})[x]$ for all $p = 2, 3, 7$ since $\mathbb{Z}/p\mathbb{Z}$ is a field.

Hence the image under ψ gives the factorization of $f(x)$ into irreducibles

$$\psi : f(x) = x \mapsto (x, x, x) = (x, 1, 1) \cdot (1, x, 1) \cdot (1, 1, x)$$

and we should look for factorization of the form $f(x) = p(x)q(x)h(x)$ in $R[x]$ where p, q, h are linear such that their image under ψ is $(x, 1, 1), (1, x, 1)$ and $(1, 1, x)$ respectively.

Let $p(x) = ax + b$, $q(x) = cx + d$ and $h(x) = ex + f$, then

$$\begin{cases} a \equiv 1 \pmod 2 \\ a \equiv 0 \pmod 3 \\ a \equiv 0 \pmod 7 \end{cases} \begin{cases} b \equiv 0 \pmod 2 \\ b \equiv 1 \pmod 3 \\ b \equiv 1 \pmod 7 \end{cases}$$
$$\begin{cases} c \equiv 0 \pmod 2 \\ c \equiv 1 \pmod 3 \\ c \equiv 0 \pmod 7 \end{cases} \begin{cases} d \equiv 1 \pmod 2 \\ d \equiv 0 \pmod 3 \\ d \equiv 1 \pmod 7 \end{cases}$$
$$\begin{cases} e \equiv 0 \pmod 2 \\ e \equiv 0 \pmod 3 \\ e \equiv 1 \pmod 7 \end{cases} \begin{cases} f \equiv 1 \pmod 2 \\ f \equiv 1 \pmod 3 \\ f \equiv 0 \pmod 7 \end{cases}$$

Again since $2, 3, 7$ are pairwise coprime, by the Chinese Remainder Theorem, the above has a unique solution $\pmod{42}$. One checks that it is given by

$$a \equiv 21, \quad b \equiv 22, \quad c \equiv 28, \quad d \equiv 15, \quad e \equiv 36, \quad f \equiv 7 \pmod{42}$$

Hence

$$f(x) = x = (21x + 22)(28x + 15)(36x + 7)$$

is the only factorization of $f(x)$ into irreducibles in $R[x]$ *up to units*, and $f(x)$ is reducible.

Problem 6. We first observe that for any continuous function, $J \subset f(J)$ implies a fixed point in a closed interval $J = [a, b]$.

Consider $g(x) = f(x) - x$. Then if $g(a) = 0$ or $g(b) = 0$ we are done. Otherwise they are of different signs, and by IVT there exists $c \in (a, b)$ with $g(c) = 0$ as required.

Now by induction we observe that since $J_n \subseteq f(J_{n-1})$, we have

$$J_2 \subseteq f(J_1)$$

$$J_3 \subseteq f(J_2) \subseteq f^2(J_1)$$

$$\vdots$$

$$J_{2026} \subseteq f(J_{2025}) \subseteq \cdots \subseteq f^{2025}(J_1)$$

and finally by assumption

$$J_1 \subseteq f(J_{2026}) \subseteq f^{2026}(J_1)$$

and we can apply the first observation.

Remark. The original intended question is to show $f^n(x_*) \in J_{n+1}$ for any $n = 1, ..., 2025$ as well, which requires much more subtle argument.

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