Basic differentiation - Implicit differentiation:

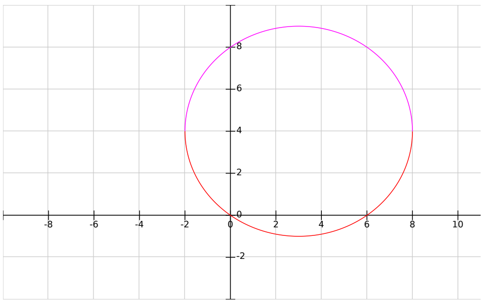
Sometimes a function is defined as the solution of an equation.

Examples.

- Consider the points (x, y) in the plane which satisfy the equation $0 = x^2 - 6x + y^2 - 8y$. The equation can be rewritten as

  $$0 = (x^2 - 6x + 3^2) + (y^2 - 8y + 4^2) - 25\,, \quad \text{so} \quad (x-3)^2 + (y-4)^2 = 5^2\,.$$

  We see the locus of points satisfying the equation is a circle of center (3,4) and radius. For x in the interval [-2,8], the circle graphically defines two functions of y for the input x: the top part of the circle, and the bottom part of the circle.

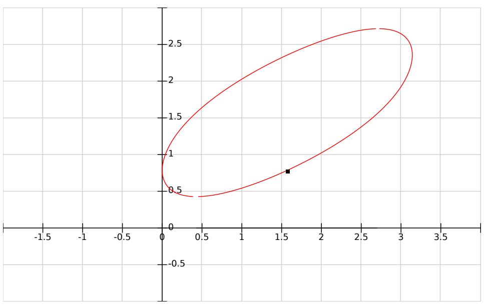
  

  For this simple example, we could use the equation $(x-3)^2 + (y-4)^2 = 5^2$ to solve for the two functions.

- The locus of points which satisfy:

  $$\cos(x - y) + \sin(y) = \sqrt{2}$$

  is an oval

  · The point $P = (\frac{\pi}{2}, \frac{\pi}{4})$ lies on the graph. What is the tangent slope there?

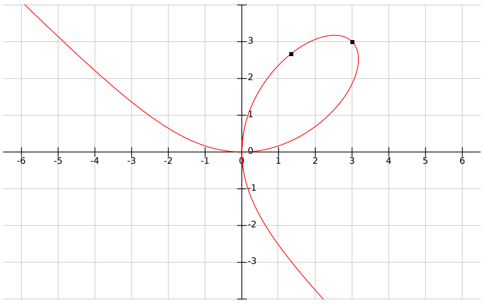
  · One can write, using arccos, a function for x in terms of y.

  · It is difficult to write an algebraic expression for y as a function of x.

- The locus of points which satisfy:

  $$x^3 + y^3 = 6xy$$

  is



- The points $P = (3,3)$ and $Q = (\frac{4}{3}, \frac{8}{3})$ lie on the graph. What is the tangent slopes at these points? Since the equation is symmetric in x and y, by symmetry we would guess the tangent slope at the point $P = (3,3)$ is -1. But what about Q?
- It is difficult to write algebraic expressions for y as a function of x and vice versa.

In the above examples, the equation does not explicitly give us an algebraic rule for y in terms of x (or vice versa$x$ in terms of y). Rather the equation gives us the graph of a function. We say the equation gives us **implicitly** (not explicitly) a function of y in terms of x (or x in terms of y).

There is a function, we just do not have a nice algebraic rule for the function.

The technique of implicit differentiation is to take the equation defining the relationship between x and y and differentiate it treating one variable as a function of the other.

Examples.

- For the circle $0 = x^2 - 6x + y^2 - 8y$, we view y as a function of x. Then differentiate to get:

$$\begin{aligned} 0 &= \frac{d}{dx}(0) = \frac{d}{dx}\left(x^2 - 6x + y^2 - 8y\right) \\ &= \frac{d}{dx}(x^2) + \frac{d}{dx}(-6x) + \frac{d}{dx}(y^2) + \frac{d}{dx}(-8y) \\ &= 2x + (-6) + (2y\frac{dy}{dx}) + (-8\frac{dy}{dx}) \\ &= (2x - 6) + (2y - 8)\frac{dy}{dx} \end{aligned}$$

We can solve for $\frac{dy}{dx}$ to get:

$$\frac{dy}{dx} = -\frac{(2x - 6)}{(2y - 8)}$$

For example, the point $P = (0,0)$ lies on the locus of points of $0 = x^2 - 6x + y^2 - y$. The tangent slope at P is:

$$\frac{dy}{dx}\Big|_{(0,0)} = -\frac{(2x - 6)}{(2y - 8)}\Big|_{(0,0)} = -\frac{(2 \cdot 0 - 6)}{(2 \cdot 0 - 8)} = -\frac{6}{8}$$

- For $\cos(x-y) \;+\; \sin(y) \;=\; \sqrt{2}$, determine the tangent slope at the point $P = (\frac{\pi}{2}, \frac{\pi}{4})$.
  - We verify P is a graph point.

$$\Big(\;\cos(x-y) \;+\; \sin(y) \;=\; \sqrt{2}\Big)\Big|_{(\frac{\pi}{2},\frac{\pi}{4})} \;=\; \cos(\frac{\pi}{2}-\frac{\pi}{4}) \;+\; \sin(\frac{\pi}{4}) \;=\; \cos(\frac{\pi}{4}) \;+\; \sin(\frac{\pi}{4}) \;=\; \sqrt{2}\;,$$

    So, $P = (\frac{\pi}{2}, \frac{\pi}{4})$ is a graph point.
  - We view y as a function for x, so $y = y(x)$. We differentiate the equation $\cos(x-y) \;+\; \sin(y) \;=\; \sqrt{2}$:

$$\begin{aligned}
\frac{d}{dx}\Big(\;\cos(x-y) \;+\; \sin(y)\Big) \;&=\; \frac{d}{dx}\,(2)\\
\frac{d}{dx}\,\cos(x-y) \;+\; \frac{d}{dx}\,\sin(y) \;&=\; 0\\
-\sin(x-y)\,\frac{d}{dx}(x-y) \;+\; \cos(y)\,\frac{d}{dx}(y) \;&=\; 0\\
-\sin(x-y)\,(1-\frac{dy}{dx}) \;+\; \cos(y)\,\frac{dy}{dx} \;&=\; 0\\
\frac{dy}{dx}\,\Big(\;\sin(x-y)+\cos(y)\Big) \;&=\; \sin(x-y)\\
\frac{dy}{dx} \;&=\; \frac{\sin(x-y)}{\sin(x-y)+\cos(y)}
\end{aligned}$$

    Therefore, the tangent slope at $P = (\frac{\pi}{2}, \frac{\pi}{4})$ is:

$$\frac{dy}{dx}\Big|_{(\frac{\pi}{2},\frac{\pi}{4})} \;=\; \frac{\sin(\frac{\pi}{2}-\frac{\pi}{4})}{\sin(\frac{\pi}{2}-\frac{\pi}{4}) \;-\; \cos(\frac{\pi}{4})} \;=\; \frac{1}{2}\;.$$

- For $x^3 + y^3 = 6xy$, determine the tangent slope at the point $Q = (\frac{4}{3}, \frac{8}{3})$.
  - We first verify the point $(\frac{4}{3}, \frac{8}{3})$ is on the curve:

$$\Big(\frac{4}{3}\Big)^3 \;+\; \Big(\frac{8}{3}\Big)^3 \;=\; \frac{64}{27} \;+\; \frac{8\cdot 64}{27} \;=\; \frac{9\cdot 64}{27} \;=\; \frac{64}{3} \qquad \text{and} \qquad 6\,\frac{4}{3}\,\frac{8}{3} \;=\; 2\cdot 3\,\frac{4}{3}\,\frac{8}{3} \;=\; \frac{64}{3}\;.$$

    So, $(\frac{4}{3}, \frac{8}{3})$ is a graph point.
  - To find the tangent slope and tangent line at Q, we treat y as function $y = y(x)$ of x and differentiate:

$$\begin{aligned}
x^3 \;+\; y^3 \;&=\; 6x\,y\\
\frac{d}{dx}\big(x^3 \;+\; y^3\big) \;&=\; \frac{d}{dx}\big(\,6x\,y\,\big)\\
3x^2 \;+\; 3y^2\,\frac{dy}{dx} \;&=\; 6\,y \;+6\,x\,\frac{dy}{dx}\\
\frac{dy}{dx} \;&=\; \frac{(3x^2 \;-\; 6y)}{(6x \;-\; 3y^2)}\\
\frac{dy}{dx}\Big|_{(\frac{4}{3},\frac{8}{3})} \;&=\; \frac{(\,3\,(\frac{4}{3})^2 \;-\; 6\,\frac{8}{3}\,)}{(\,6\,\frac{4}{3} \;-\; 3\,(\frac{8}{3})^2\,)} \;=\; \frac{4}{5}
\end{aligned}$$

    The tangent slope at $Q = (\frac{4}{3}, \frac{8}{3})$ is $\frac{4}{5}$.

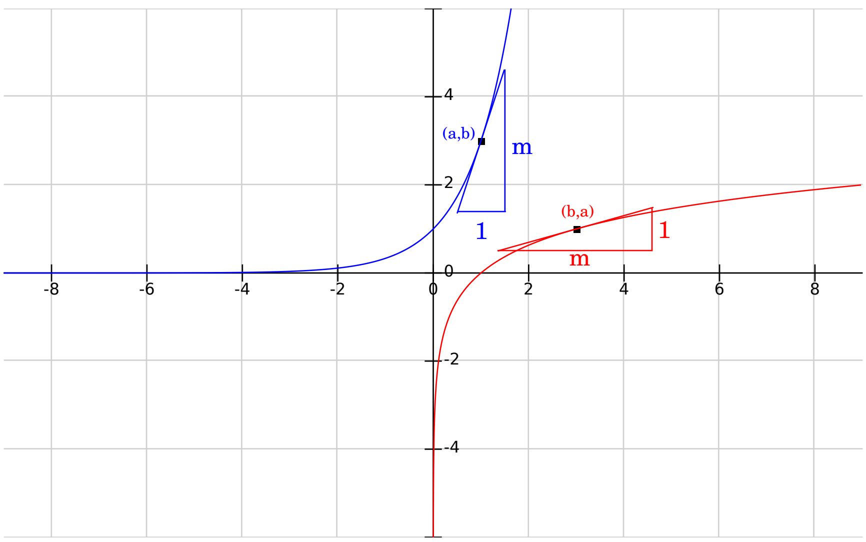
    The tangent line is:

$$y \;=\; \frac{4}{5}\,(\,x \;-\; \frac{4}{3}\,) \;+\; \frac{8}{3}\;.$$

Suppose f is function with an interval domain $\mathcal{D}$ and which is one-to-one and differentiable on the domain. The one-to-one property means there will be an inverse function g. The functions f and g 'undo each other'.

$$b = f(a) \iff g(b) = a$$

point (a, b) is on the graph of f	$\iff$	point (b, a) is on the graph of g
tangent slope at (a, b) is m	$\iff$	tangent slope at (b, a) is $\frac{1}{m}$



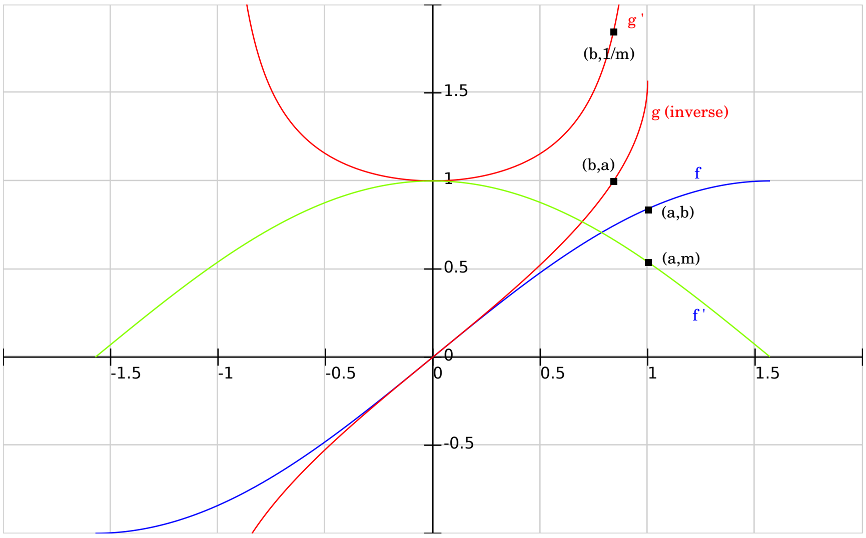
- Derivative of inverse: Suppose f is is one-to-one and differentiable and g is the inverse function. Suppose $b = f(a)$ (so $g(b) = a$), and $f'(a) \neq 0$. The g is differentiable at b, and

$$g'(f(a)) = \frac{1}{f'(a)}.$$

Examples.

- The arcsine function $x = (\arcsin(y))$ is the inverse of the sine function $y = \sin(x)$. The derivative of arcsin is:

$$(\arcsin(y))'\Big|_{y=\sin(x)} = \frac{1}{(\sin(x))'} = \frac{1}{\cos(x)} = \frac{1}{\sqrt{1-(\sin(x))^2}} = \frac{1}{\sqrt{1-y^2}}, \quad \text{so}$$
$$(\arcsin(y))' = \frac{1}{\sqrt{1-y^2}}\,.$$



- The arctangent function $x = (\arctan(y))$ is the inverse of the tangent function $y = \tan(x)$. The derivative of arctan is:

$$(\arctan(y))'\Big|_{y=\tan(x)} = \frac{1}{(\tan(x))'} = \frac{1}{\left(\frac{1}{(\cos(x))^2}\right)} = \frac{1}{\left(\frac{(\cos(x))^2+(\sin(x))^2}{(\cos(x))^2}\right)}$$
$$= \frac{1}{(1\ +\ (\tan(x))^2)} = \frac{1}{(1\ +\ y^2)}, \quad \text{so}$$
$$(\arctan(y))' = \frac{1}{(1\ +\ y^2)}\,.$$

- The natural logarithm ln is the inverse of the exponential function $y = e^x$: $\quad x = \ln(y)$. The derivative of ln is:

$$(\ln(y))'\Big|_{y=e^x} = \frac{1}{(e^x)'} = \frac{1}{e^x} = e^{-x} = \frac{1}{y}, \quad \text{so}$$

$$(\ln(y))' = \frac{1}{y}\,.$$