Review

1 Functions.

1.1 Sets.

Recall a set is a collection of objects called the elements of the set.
Examples:

- The collection $\{ 0, 1, 2, 3, \ldots \}$ of zero and the positive integers is a set. For easy referral we call it the set of natural integers, and denote it in writing as $\mathbb{N}$.
- The collection $\{ 0, \pm 1, \pm 2, \pm 3, \ldots \}$ of all integers is a set. It is denoted it in writing as $\mathbb{Z}$.
- The collection $\{ \text{fractions } \frac{p}{q} \mid p,\ q \ \text{ integers } \}$ of rational numbers is denoted in writing as $\mathbb{Q}$.

- The collection of (real) decimal numbers such as

$$\frac{13}{11} = 1.181818\ldots\ ,\ \sqrt{2} = 1.4142135\ldots\ ,\ \pi = 3.14159\ldots$$

  is the set of real numbers. It is denoted as $\mathbb{R}$.

- A non-numerical example of a set is the collection of McDonalds menu items

  $\mathcal{M} = \{ \text{chicken-nuggets, big-mac, fish sandwich, } \ldots \text{ , mcflurry} \}$

Functions are used to express relationships among variables.

1.2 Ingredients of a function.

- Input set D
- Output set C
- Rule f

A function f is a rule which assigns to each element of the input set D, an element $f(x)$ in the output set:

$$x \in D \xrightarrow{\text{input}} \text{rule f} \xrightarrow{\text{output}} f(x) \in C$$

Notation:

- The input set is call the **domain**.
- The output set is call the **codomain**.
  The precise set of outputs is called the **range** of the function.
- The output element $f(x)$ is called the **value** of the function f at input x.

Example: Take

- Domain (input set) to be $\mathcal{M}$, the set of McDonald menu items.
- Codomain (output set) to be the set $\mathbb{N}$ of natural integers.
- function to be the price function P which is the price (in cents) of a menu item

$$x \in \mathcal{M} \xRightarrow{\text{input}} \text{Price function P} \xRightarrow{\text{output}} P(x) \in \mathbb{N}$$

$$\text{chicken-nugget} \xrightarrow{\text{price}} P(\text{chicken-nugget})$$

So, McDonalds' menu table is a function!

- **Verbally** description in words
- **Numerically** table of values
- **Algebraically** formula
- **Graphically** by a graph

Verbal example: Take:

· Domain to be the alphabet $\mathcal{A} = \{\ A,\ B,\ C,\ \ldots\ ,\ Y,\ Z\ \}$

· Codomain to be the natural integers $\mathbb{N}$.

· rule p to be the position in the alphabet; so,

$$p(B) = 2, \quad p(L) = 12, \quad p(Y) = 25, \quad \text{etc}$$

Question: What is the range set (precise set of values) of the position function?

Numerical/tabular example: Take:

· **Domain** $\mathcal{D}$ to be the set URL of webnames.
For example www.facebook.com $\in \mathcal{D}$ (URL).

· **Codomain** to be the set $\mathcal{I}$ of all possible internet IP–addresses:

aaa.bbb.ccc.ddd where each tiple is between 0 and $255 = 2^8 - 1$

· **rule** f to be the ‘internet domain function’ which takes a URL x and gives the IP-number of x. For example:

$$f(\text{www.facebook.com}) = 173.252.91.4$$

$$f(\text{www.ust.hk}) = 143.89.14.2$$

Each time we enter a URL into a browser, it goes to the internet to lookup the IP-address of the URL and then retrieves information stored on the machine with IP-address $f(\text{URL})$.

One way for hackers to disrupt the internet is by attacking the internet machines which are the repository for the function/table of URL to IP-addresses.

Examples of functions given algebraically: Take:

(1) $f \ : \ \mathbb{R} \longrightarrow \mathbb{R}$

$$x \xrightarrow{\text{f}} y = f(x) = x^2 - 4$$

(2) $g \ : \ \mathbb{R} \longrightarrow \mathbb{R}$

$$x \xrightarrow{\text{g}} y = g(x) = \frac{1}{1+x^2}$$

(3) Consider the rule:

$$x \xrightarrow{\text{h}} y = h(x) = \frac{1}{x-3}$$

Since division by 0 is not allowed, the rule must avoid $x = 3$, so the domain must be the set of numbers NOT equal to 3. This set can be written in several ways such as:

$$\{ \ x \in \mathbb{R} \mid x \neq 3 \ \} \quad \text{or} \quad \mathbb{R} \ - \ \{ 3 \}$$

Example of a function given graphically:
Graph of the closing stock price of FaceBook during 2012-2014:
Domain is the set of days. Codomain is $\mathbb{R}$.

3 Vertical line test

Question: When is the set of points in the plane; for example the line $x + y = 5$, or the circle $(x-2)^2 + (y-2)^2 = 5^2$ the graph of a function?

Vertical Line Test: A set S in the plane is the graph of a function if each vertical line meets S in **at most one point**.

(1) The graph of the line $x + y = 5$ is the graph of a function. The function can be given algebraically as $y = f(x) = 5 - x$.

(2) The graph of the circle $(x-2)^2 + (y-2)^2 = 5^2$ is not the graph of a function. Vertical lines $x = b$ for $-3 < b < 5$ meet the circle

in two points. When we solve for y in terms of x we get

$$\begin{aligned} (y-2)^2 &= 25-(x-2)^2 \\ (y-2) &= \pm\sqrt{25-(x-2)^2} \\ y &= 2 \pm \sqrt{25-(x-2)^2} \end{aligned}$$

(3) The graph of the parabola $y - x^2 = 5$ is the graph of a function. The function can be give algebraically as $y = h(x) = 5 + x^2$.

(4) The graph of the parabola $y^2 - x = 5$ is the not the graph of a function. When we solve for y in terms of x we get

$$\begin{aligned} y^2 &= 25 - x \\ y &= \pm\sqrt{25-x} \end{aligned}$$

Basic Functions

4 Basic Functions.

Some basic functions given by a formula rule are:

- Linear functions: $y = f(x) = mx + b$
- Polynomials:
  $$P(x) = a_m x^m + a_m x^m + \cdots + a_2 x^2 + a_1 x + a_0$$
- Rational functions:
  $$r(x) = \frac{P(x)}{Q(x)}, \text{ where } P(x) \text{ and } Q(x) \text{ are polynomials}$$
- Power functions: Functions of the form $f(x) =$
  $$\sqrt{x}, \quad x^{\frac{1}{3}}, \quad x^{-\frac{5}{7}}, \quad \ldots$$
- Trigonometric functions: $\sin(x)$, $\cos(x)$, $\tan(x)$, ...
- Exponential functions: 10^x, 2^x,, 3^x, ...

4.1 Linear functions.

$f(x) = mx + b$

- Very simple rule (easy to compute)
- Graph is a line:

  · slope is m

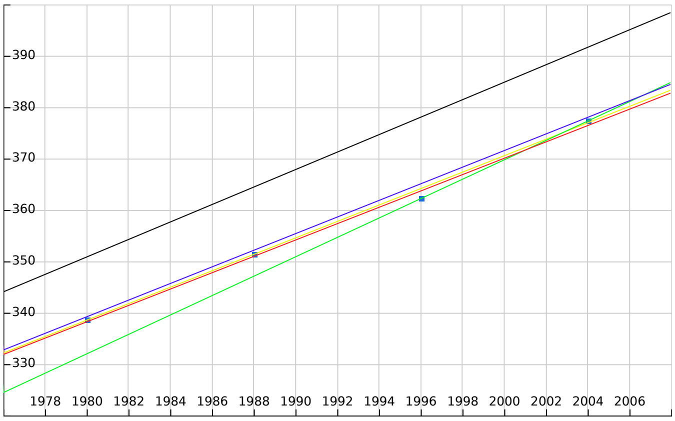
  · point $(0, b)$ is on graph, i.e., y-intercept is b
- Often used to approximate more complicated functions.

Example: CO_2 levels in the atmosphere

Year	CO_2 level (parts/million)	graph point
1980	338.7	$p_1 = (1980, 338.7)$
1988	351.5	$p_2 = (1988, 351.5)$
1996	362.4	$p_3 = (1996, 362.4)$
2004	377.5	$p_4 = (2004, 377.5)$

The four points do not lie on a line: The 4 input years increase by 8 years each time, but the 3 differences in the CO_2 level increased by 12.8, 10.9, and 15.1 which changed from 8-year period to 8-year period to 8-year period.

Graphs of the linear function $L(x) = mx + b$ for various slopes m and y-intercept b



The best "least squares" line is the choice of slope m and y-intercept b so the function

$$y = L(x) = m\,x + b$$

has the property that for our four table values $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (x_3, y_3)$, and $p_4 = (x_4, y_4)$, the 'sum of the squared differences':

$$\begin{aligned} \text{'Error'} &= \text{'Error at point } p_1\text{'} + \text{'Error at point } p_2\text{'} \\ &\quad + \text{'Error at point } p_3\text{'} + \text{'Error at point } p_4\text{'} \\ &= \big((\,m\,x_1 + b\,) - y_1\big)^2 + \big((\,m\,x_2 + b\,) - y_2\big)^2 \\ &\quad + \big((\,m\,x_3 + b\,) - y_3\big)^2 + \big((\,m\,x_4 + b\,) - y_4\big)^2 \end{aligned}$$

is the smallest possible.

Choice of slope m and intercept b	Error at point p_1	Error at point p_2	Error at point p_3	Error at point p_4	Sum of errors at 4 points
(1.6000 , -2829.3) (yellow)	0.00	0.00	3.61	0.16	3.77
(1.8875 , -3405.1) (green)	42.25	17.64	0.00	0.00	59.89
(1.6167 , -2862.3) (blue)	0.00	0.02	4.69	0.00	4.71
(1.7000 , -3015.0) (black)	151.29	171.61	249.64	204.49	777.03
(1.5912 , -2812.2) (red)	0.13	0.19	1.95	0.95	3.22

Calculus can be used to find the 'best' choice of slope m and intercept b. It is: $m = 1.5912$, $b = -2812.24$ and

$$\begin{aligned} y &= L(x) = 1.5912\,x - 2812.24 \\ &= 1.5912\,(x - 1980) + 338.4\,. \end{aligned}$$

The example **best least squares prediction** for the CO_2 levels in 2020 is

$$\begin{aligned} L(2020) &= 1.5912\,(2020 - 1980) + 338.4 \\ &= 402.05 \ \text{parts/million} \end{aligned}$$

4.2 Exponential functions.

An exponential function is defined in terms of a positive base b. For example, base 10. We know how to compute:

· Integer powers of 10;

$$10^3 \text{ (thousand)}, \quad 10^6 \text{ (million)}, \quad 10^{-9} \text{ (nano)}$$

· Fractions powers of 10:

$$\sqrt{10} = 3.1622\ldots, \quad 10^{\frac{1}{4}} = 1.7782\ldots$$

It is possible to define the power 10^x for any number x.
For any positive base b, it is posible to define the power b^x. The rule which takes input x and gives output b^x is the exponential function. The function/rule is written as

$$\exp_b \ .$$

For example, some calculators have a button label $\exp_{10}$.

Properties of the exponential functions are:

(i) $\exp_b(1) \;=\; b$

(ii) $\exp_b(x+y) \;=\; \exp_b(x)\;\exp_b(y)$ (turns addition into multiplication)

(iii) $\exp_b$ is a continuous function

(iv) $(b^x)^y \;=\; b^{xy}$

Property (iii) means if we have a sequence of inputs $x_1, x_2, x_3, \ldots$ which "converge" to a number x, then the sequence of outputs $b^{x_1}, b^{x_2}, b^{x_3}, \ldots$ converge to the output b^x. This property is very inportant. There are 'useless' functions which satisfy (i) and (ii) but not (iii).

5 One-to-one and onto functions.

Two important properties which a function may or may not have are:

one-to-one, and **onto**

5.1 One-to-one

A function f is one-to-one if

different inputs produce different outputs

We express this mathematically as saying if inputs a and b are not equal, then the outputs $f(a)$ and $f(b)$ are not equal.

$$a,\, b\; \in \mathcal{D} \text{ (domain), and } a \neq b \implies f(a) \neq f(b) \text{ (in codomain)}$$

This is the same as:

$$a,\, b\; \in \mathcal{D}, \text{ and } f(a) = f(b) \implies a = b$$

Examples:

- A linear function
$$L(x) = mx + b$$
with slope $m \neq 0$ is one-to-one. Suppose x_1, and x_2 are two inputs which give the same output: $L(x_1) = L(x_2)$. Then
$$\begin{aligned} L(x_1) &= L(x_2) \\ m\,x_1 \ + b &= m\,x_2 \ + b \quad \text{, so} \\ m\,x_1 &= m\,x_2 \quad \text{, now divide by } m \neq 0 \\ x_1 &= x_2 \ . \end{aligned}$$
Conclude a linear function $L(x) = mx + b$ with non-zero slope is one-to-one.

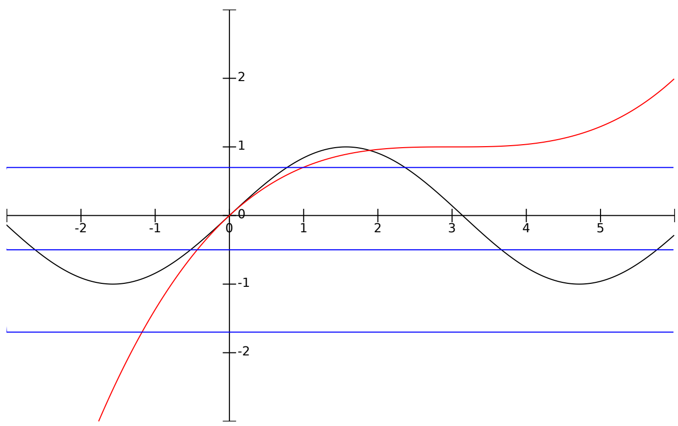
- A linear function $L(x) = 0\,x + b$ with slope 0, is a constant function. Such functions are not one-to-one.

Horizontal line test for graph functions in the plane:

If a function f is described as a graph in the plane, then f is one-to-one precisely when each horizontal line in the plane meets the graph in at most one point.

If a horizontal line $y = b$ meets the graph in two or more points $p_1 = (x_1, b)$ and $p_2 = (x_1, b)$, then $f(x_1) = f(x_2)$ with $x_1 \neq x_2$, so the function is not one-to-one.

Example: $\sin(x)$ is not one-to-one, $(\frac{x}{3} - 1)^3 + 1$ is one-to-one



5.2 Onto

The range of a function is the complete set of its values.

Examples:

1. The sin function has domain $\mathbb{R}$. The complete set of its values is all numbers between -1 and 1.
2. The function $y(x) = x^2$ has domain $\mathbb{R}$. The complete set of its values is numbers $y \geq 0$.
3. The function $y(x) = x^3$ has domain $\mathbb{R}$. The complete set of its values is $\mathbb{R}$

For a particular function f with domain $\mathcal{D}$, we usually have some choice in what we call the codomain. In each of the examples, above, we could take the codomain to be $\mathbb{R}$, a larger set than the range.

A function f with domain $\mathcal{D}$ is **onto** a codomain $\mathcal{C}$ if the codomain equals the range of the function.

Examples:

1. Consider the sin function, with domain $\mathbb{R}$.
   - If we take the codomain to be $\mathbb{R}$, then the sin function is not onto the codomain.
   - If we instead take the codomain to be $\mathcal{C} = \{ -1 \leq y \leq 1 \}$, then the function is onto the codomain.
2. Consider the function $y(x) = x^2$, with domain $\mathbb{R}$.
   - If we take the codomain to be $\mathbb{R}$, then the function is not onto the codomain.
   - If we instead take the codomain to be $\mathcal{C} = \{ 0 \leq y \}$, then the function is onto the codomain.

3. Suppose $\mathcal{D}$ is a collection of at most 300 people. Take the set $\mathcal{C}$ to be the the dates of the year, so

$$\mathcal{C} \;=\; \{\text{ Jan 01, Jan 02, } \ldots \text{ , Dec 31 }\}$$

Let $B \,:\, \mathcal{D} \longrightarrow \mathcal{C}$, be the rule which takes input (person) x to their birthday $B(x)$.

Question: For the (365 element) codomain $\mathcal{C}$, why cannot the function B be onto?

A function $f : \mathcal{D} \longrightarrow \mathcal{C}$ is onto if:

$$\text{For any } y \in \mathcal{C}, \text{ there is a } a \in \mathcal{D}, \text{ with } y = f(a)$$

In words, any element of the codomain appears as a output/value of the function.

6 Inverse functions

When a function $f : \mathcal{D} \longrightarrow \mathcal{C}$ is both one-to-one and onto, then one can "reverse" the function to get a function $g : \mathcal{C} \longrightarrow \mathcal{D}$. The roles of the domain and codomain have are reversed, and we think of the the process g as undoing the function f.

Examples:

1. A linear function $L(x) = mx + b$ from the domain $\mathbb{R}$ to the codomain $\mathbb{R}$, with non-zero slope m, is one-to-one and onto. The rerevse is obtained by solving for x in terms of y. It is the rule

$$R(y) \;=\; \frac{1}{m}\,(\,y \;-\; b\,)\;.$$

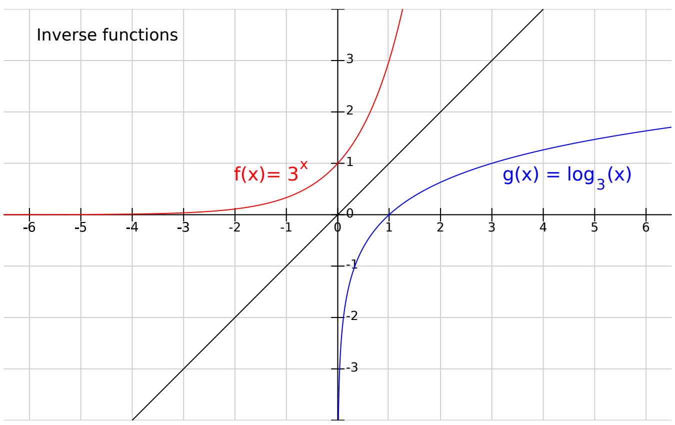
2. The rule $S(x) = x^2$ from the domain $\mathbb{R}$ to the codomain $\mathbb{R}$ is neither one-to-one, nor onto:
   - Not one-to-one since $f(x) = f(-x)$, so different inputs can produce same output.
   - Not onto since the outputs (x^2) are always ≥ 0, and so cannot take on any of the negative numbers in the codomain.
3. The same rule $S(x) = x^2$ from the domain $\mathbb{R}_{\geq 0}$ to the codomain $\mathbb{R}_{\geq 0}$ is both one-to-one and onto. The reverse function is the square root function:

$$R(y) = \sqrt{y}\,, \quad \text{the positive square root of } y.$$

6.1 Logarithm

When $b > 1$, the exponential function $\exp_b : \mathbb{R} \longrightarrow \mathbb{R}_{>0}$ (note: $\mathbb{R}_{>0}$ means the positive numbers) is one-to-one and onto (the positive numbers). The inverse function is called the logarithm to base b, and denoted $\log_b$.

Example:



If the domain and codomain sets of a one-one and onto function $f\ :\ \mathbb{D} \longrightarrow \mathbb{C}$ are sets of real numbers, the graph of the inverse function R is obtained from the graph of f by swapping coordinates

$$(a , b) \longleftrightarrow (b , a) .$$

Geometrically the graph of the ineverse function R is obtained from the graph of f by reflection across the $y = x$ line.

6.2 Logarithm formula between different bases

The general formula relating the functions $\log_b$ and $\log_a$ is:

$$\log_a(x) \ = \ \frac{\log_b(x)}{\log_b(a)} \ .$$

7 Function composition

7.1 Definition of function composition

Suppose:

- f is a function with domain $\mathcal{D}$ and codomain $\mathcal{C}$, so

$$\mathcal{D} \xrightarrow{\ f\ } \mathcal{C}\ ,$$

  and

- g is a function with domain $\mathcal{C}$ and codomain $\mathcal{B}$, so

$$\mathcal{C} \xrightarrow{\ g\ } \mathcal{B}\ .$$

We can form the **composite function** $g \circ f$, which is a function with domain $\mathcal{D}$ and codomain $\mathcal{B}$

$$\text{input } x \in \mathcal{D} \xrightarrow{\quad f \quad} f(x) \in \mathcal{C} \xrightarrow{\quad g \quad} \text{output } \ g(f(x)) \in \mathcal{B}$$

Example: The set of real number greater than or equal to zero is denoted $\mathbb{R}_{\geq 0}$. Take

$$\mathbb{R}_{\geq 0} \xrightarrow{\quad b(t) = \sqrt{t} \quad} \mathbb{R}_{\geq 0} \quad \text{(which is inside } \mathbb{R}\text{)}$$

$$\mathbb{R}_{\geq 0} \xrightarrow{\quad c(u) = \frac{1}{1+u} \quad} \mathbb{R}_{\geq 0}$$

The two functions $b \circ c$ and $c \circ b$ both make sense:

$$(b \circ c)\,(u) \;=\; b(c(u)) \;=\; b(\frac{1}{1+u}) \;=\; \sqrt{\frac{1}{1+u}}$$

is a function from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$

$$(c \circ b)\,(t) \;=\; c(b(t)) \;=\; c(\sqrt{t}\,) \;=\; \frac{1}{1+\sqrt{t}}$$

is a function from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$

7.2 Associativity of composition.

If a, b, and c are three functions, the two functions

$$(a \circ b) \circ c \quad \text{and} \quad a \circ (b \circ c)$$

are equal. Their value at an input u is:

$$a(b(c(u))) \ .$$

8 Basic changes to the graph of a function

Suppose $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is a function with domain and range $\mathbb{R}$, and a, b, c are (fixed) numbers. We can consider the functions:

$$a\, f(x) \ , \quad f(bx) \ , \quad \text{and } f(x-c) \ .$$

The relation of the graphs of these three functions to the graph of the original function f is the following:

- The graph of $af(x)$ is obtained by **vertically scaling** the graph of $f(x)$ by a factor of a.
- Assume $b \neq 0$. The graph of $f(bx)$ is obtained by **horizontally scaling** the graph of $f(x)$ by a factor of $\frac{1}{b}$.
- The graph of $f(x-c)$ is obtained by a **horizontal rightward translation** of the graph of $f(x)$ by c.

Example:

We take $f(x) = x^3 - 5x + 9$. The graphs of

$$(0.5)f(x)\ , \qquad f(\frac{x}{0.8})\ , \quad \text{and} \quad f(x-2)$$

are: