9 Definition of when a function has a limit.

Suppose $\mathcal{D}$ is an interval, and

f is a function whose domain is $\mathcal{D}$ with the possible exception of an interior point b.

For example, for the function $y = x^2$, and $P = (b, b^2)$, the secant slope of the line P and $Q = (x, x^2)$ is

$$m_P(x) = \frac{x^2 - b^2}{x - b}$$

In this algebraic expression (m_P) be we must exclude b – division by zero is not allowed.

9.1

We say the function f has a limit L as $x \to b$ if:

1st Intuition formulation of limit: We can assure the output values $f(x)$ are close to L by taking the input x to be close to but not equal to b.

Examples.

- $\lim\limits_{x \to b} x^3 = b^3$.

  Our intuition says if we take x near to b, then x^3 should be near to b^3.

- For the function $y = x^2$, since the secant slope of $P = (b, b^2)$ and $Q = (x, x^2)$ is $\frac{x^2 - b^2}{x - b}$, and

$$m_P(x) = \frac{x^2 - b^2}{x - b} = x + b \,,$$

  if we now take the limit of the secant slope as $x \to b$ we get:

$$\lim_{x \to b} \frac{x^2 - b^2}{x - b} = \lim_{x \to b} (x + b) = 2b \,.$$

  This limit is the tangent slope to the graph at the point P.

9.2

2nd More quantitative formulation of limit:

· If we take x near to (but not equal to) b; so $0 < |x-b|$ is small,

· then $f(x)$ will be near to L, that is $|f(x)-L|$ is small.

Example. We use this 2nd definition of limit to show $\lim_{x\to b} \sqrt{x} = \sqrt{b}$.

The limit value here is $L = \sqrt{b}$. We have

$$|\sqrt{x} - \sqrt{b}| = |\sqrt{x} - \sqrt{b}| \frac{|\sqrt{x} + \sqrt{b}|}{|\sqrt{x} + \sqrt{b}|} = |x - b| \frac{1}{|\sqrt{x} + \sqrt{b}|}$$

Therefore, if we make $|x-b|$ small, the quantity $|\sqrt{x} - \sqrt{b}| = \frac{|x-b|}{|\sqrt{x}+\sqrt{b}|}$ will be small too.

9.3

Formulation of limit in a quantitative manner:

3rd Quantitative formulation of limit:

· Given a **challenge** to make the quantity $|f(x)-L|$ small, say smaller than some **tolerance** T,

· we can find a **'tolerance-reply'** positive number R with the property that

$$0 < |x-b| < R \overset{\text{implies}}{\Longrightarrow} |f(x)-L| < T .$$

Examples. We use the quantitative definition of limit to show:

- $\lim_{x\to b} \sqrt{x} = \sqrt{b}$.

  We will asssume $b \neq 0$. We calculated above that $|\sqrt{x} - \sqrt{b}| = \frac{|x-b|}{|\sqrt{x}+\sqrt{b}|}$. Since $\sqrt{b} \leq (\sqrt{x}+\sqrt{b})$, we have $|\sqrt{x} - \sqrt{b}| \leq \frac{|x-b|}{\sqrt{b}}$. Given a challenge to make $|\sqrt{x} - \sqrt{b}| < T$, we see we can do so by taking $|x-b| < R = T\sqrt{b}$.

- $\lim_{x \to b} x^2 = x^2.$

We will asssume $b > 0$. We algebraically manipulate $|x^2 - b^2|$ to get

$$\begin{aligned} |x^2 - b^2| &= |(x-b)(x+b)| \\ &= |x-b|\,|x+b| \end{aligned}$$

If we are presented with a tolerance $T > 0$ and challenged to make $|x^2 - b^2| < T$, we can do so by insuring two things (since $b > 0$):

(i) Make $|x - b|$ less than $\frac{T}{3b}$, and

(ii) make $|x + b|$ less than $3b$.

The first is the requirement $|x - b| < \frac{T}{3b}$. The second means (since $b > 0$) that

$-3b < x + b < 3b$ so substract $2b$ to get $-3b - 2b < x - b < 3b - 2b = b$

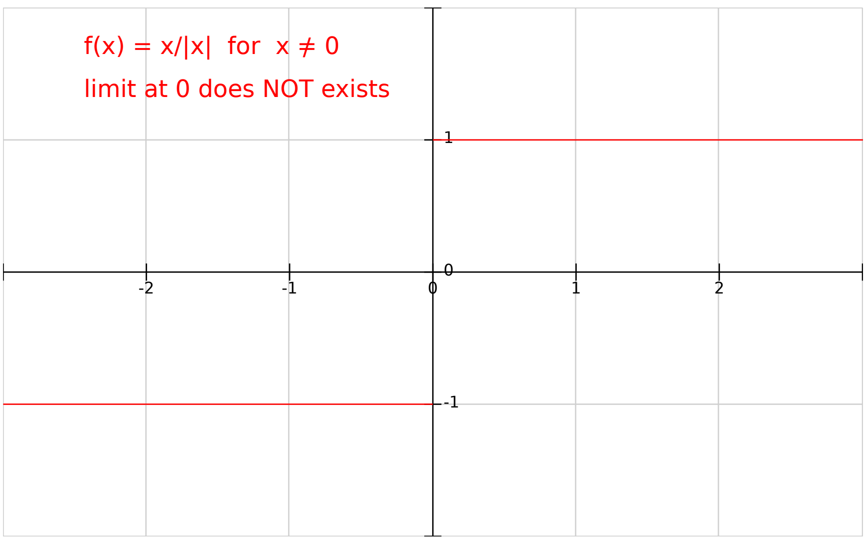
Now, $-5b < (x - b) < b$ will be true when $|x - b| < b$.

We can make BOTH $|x - b| < \frac{T}{3b}$ and $|x + b| < 3b$ true by taking $|x - b| < \frac{T}{3b}$ and $|x - b| < b$. This means take $|x - b|$ less than BOTH $\frac{T}{3b}$ and b. So our reply R to the challenge T (to make $|x^2 - b^2| < T$) is to take

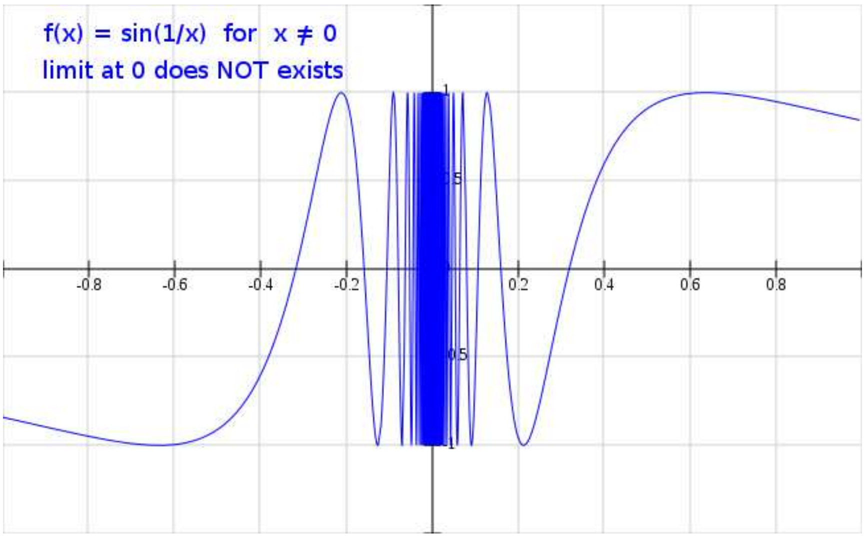
$$\begin{aligned} |x - b| &< \text{minimum of } \frac{T}{3b} \text{ and } b \\ R &= \text{minimum of } \frac{T}{3b} \text{ and } b \end{aligned}$$

10 Examples when the limit does not exists.

- The function $\frac{|x|}{x}$, which is defined for $x \neq 0$ does not have a limit as $x \to 0$.



- The function $\sin(\frac{1}{x})$, which is defined for $x \neq 0$ does not have a limit as $x \to 0$.



11 Rules for calculating limits.

Suppose $\mathcal{D}$ is an interval, $a \in \mathcal{D}$, and f and g are two functions with

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M \ .$$

Then,

- Sum rule: $\lim_{x \to a} (f + g)(x) = L + M$
- Product rule: $\lim_{x \to a} (f\, g)(x) = L\, M$

  If we take g to be a constant function $g(x) = c$, we get $\lim_{x \to a} (cf)(x) = c\, L$.
- Quotient rule: If $M \neq 0$, then $\lim_{x \to a} (\frac{f}{g})(x) = \frac{L}{M}$.

Examples

- If $p(x) = c_r x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0$ is a polynomial function, then: $\lim_{x\to a} p(x) \;=\; p(a)$.

  The reasoning is:

  · $\lim_{x\to a} x \;=\; a$. Applying the product rule, we get $\lim_{x\to a} x^2 \;=\; a^2$, and in general $\lim_{x\to a} x^k \;=\; a^k$.

  · Apply product rule again to get $\lim_{x\to a} c_k\, x^k \;=\; c_k\, a^k$.

  · Apply sum rule repeatedly to get

$$\lim_{x\to a} \left(c_r x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0 \right) \;=\; \left(c_r a^r + c_{r-1}a^{r-1} + \cdots + c_1 a + c_0 \right).$$

- If $f(x) = \frac{p(x)}{q(x)} = \frac{c_r x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0}{d_s x^s + d_{s-1}x^{s-1} + \cdots + d_1 x + d_0}$ is a rational function, and $q(a) \neq 0$, then $\lim_{x\to a} \frac{p(x)}{q(x)} \;=\; \frac{p(a)}{q(a)}$. The reasoning is:

  · By the 1st example, $\lim_{x\to a} p(x) \;=\; p(a)$, and $\lim_{x\to a} q(x) \;=\; q(a)$.

  · Since $q(a) \neq 0$, we can apply the quotient rule.

Composition rule for limits.

Suppose f and g are functions whose composition $f \circ g$ makes sense. If $\lim_{x\to a} g(x) \;=\; b$, and $\lim_{y\to b} f(y) \;=\; L$, then

$$\lim_{x\to a} (f \circ g)(x) \;=\; L$$

Example Find $\lim_{x\to 4} \sqrt{x^2+1}$.

We have $\lim_{x\to 4} \sqrt{x^2+1} \;=\; \sqrt{17}$. The reasoning is:

· The function $\sqrt{x^2+1}$, is the composition of the inside function $g(x) = x^2 + 1$ and the outside function $g(y) = \sqrt{y}$.

· $\lim_{x\to 4} (x^2+1) \;=\; 4^2 + 1 \;=\; 17$, and $\lim_{y\to 17} \sqrt{y} \;=\; \sqrt{17}$.

The Squeeze Theorem for limits.

Suppose a function g is 'squeezed' between two other functions f and h near the point a in the sense that

$$f(x) \leq g(x) \leq h(x) \quad \text{for } x \text{ near (but not equal to) } a.$$

If both $\lim_{x\to a} f(x) = L$, and $\lim_{x\to a} h(x) = L$, then

$$\lim_{x\to a} g(x) = L$$

Example

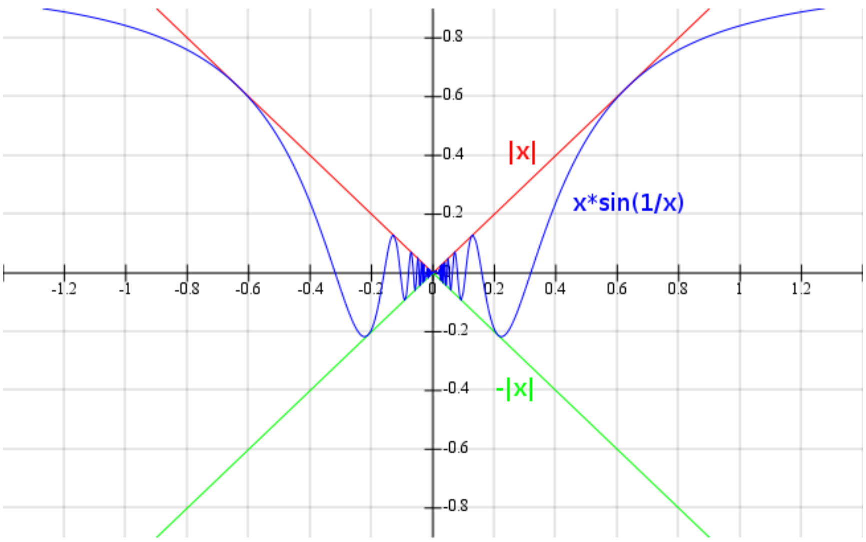
The function $g(x) = x\sin(\frac{1}{x})$ is not define at $x = 0$. Determine $\lim_{x\to 0} x\sin(\frac{1}{x})$.

The limit is 0. The reasoning is:

- Consider the two functions $f(x) = -|x|$ and $h(x) = |x|$. Since $|\sin(\cdot)| \leq 1$, the function $x\sin(\frac{1}{x})$ is squeezed between $-|x|$ below and $|x|$ above.
- $\lim_{x\to 0} -|x| = 0$ and $\lim_{x\to 0} |x| = 0$

Therefore $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$.

The function $x\sin(\frac{1}{x})$ (blue) is squeezed between the functions $-|x|$ and $|x|$ (green and red).



$$\lim_{x\to 0} -|x| = 0 \text{ and } \lim_{x\to 0} |x| = 0 \implies \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$$