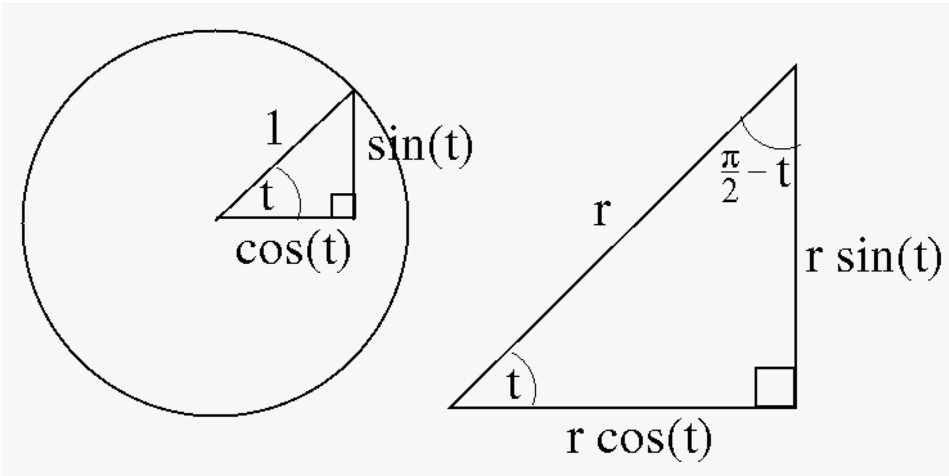
Review: trigonometry

8 Review of trigonometry

8.1 Definition of sine and cosine

Recall we measure angles in terms of degrees or radians:

$$360^\circ \;=\; 2\pi \text{ radians}$$



8.2 Basic facts:

1. The point $(\cos(t), \sin(t))$ lies on the unit circle with center $(0, 0)$, so $(\cos(t))^2 + (\sin(t))^2 \;=\; 1$.

2. From the picture $\cos(\frac{\pi}{2} \;-\; t\,) \;=\; \sin(t)$.

3. For any angle t, if we add 2π (radians) more, be have done completely around, so:

$$\sin(t + 2\pi) \;=\; \sin(t) \;\;, \quad \text{and} \quad \cos(t + 2\pi) \;=\; \cos(t)\;.$$

We say sin and cos are periodic with period 2π.

4. cos is an even function, and sin is an odd function.

5. Combining facts 2 and 4, we get

$$\cos(\,t \;-\; \frac{\pi}{2}) \;=\; \cos(\frac{\pi}{2} \;-\; t\,) \;=\; \sin(t)$$

Therefore, the graph of sin is rightward shift of the graph of cos by the amount $\frac{\pi}{2}$.

6. The tangent, secant and cosecant functions are defined as:

$$\tan(t) \;=\; \frac{\sin(t)}{\cos(t)} \;\;, \quad \sec(t) \;=\; \frac{1}{\cos(t)} \;\;, \quad \text{and} \quad \text{cosec}(t) \;=\; \frac{1}{\sin(t)}\;.$$

The tangent function $\tan(t)$ measures the slope of the angle t, and it is an odd function and periodic with period π (180 degrees).

7. Two important trigonometric identities which we use later to compute the derivative of the sin and cos functions are:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

9 Inverse trigonometic functions

9.1 Intervals of real numbers

If $a < b$ are two real numbers the set of numbers between a and b is called the interval between a and b. To be more precise, we use the notations:

$$[a, b] = \{ x \in \mathbb{R} \mid a \le x \le b \}$$ the **closed** interval between a and b

$$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$$ the **open** interval between a and b

$$[a, b) = \{ x \in \mathbb{R} \mid a \le x < b \}$$ the **half open/closed** interval between a and b

$$(a, b] = \{ x \in \mathbb{R} \mid a < x \le b \}$$ the **half open/closed** interval between a and b

for the 4 types of intervals of numbers between a and b. We can also allow the numbers a and b to be infinity:

$$(-\infty, b] = \{ x \in \mathbb{R} \mid x \le b \}, \text{ etc }, \quad (a, \infty) = \{ x \in \mathbb{R} \mid a < x \}$$

The usual domains for the functions cos and sin is the entire set of (real) numbers $\mathbb{R}$. The range of both cos, and sin is:

$$\text{range} = \mathcal{C} = \{-1 \le y \le 1\} = [-1, 1] .$$

But, cos and sin are not one-to-one functions on $\mathbb{R}$. If we restrict the domain of sin to be the interval $\mathcal{D}' = [-\frac{\pi}{2}, \frac{\pi}{2}]$, where sin is an increasing function, then

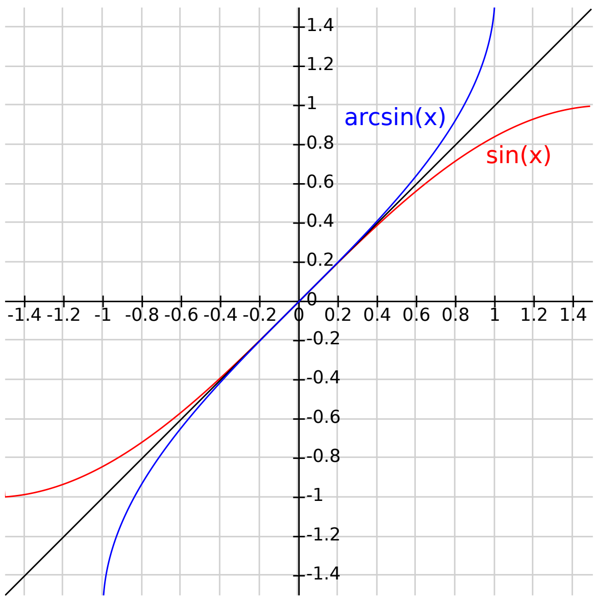
$$[-\frac{\pi}{2}, \frac{\pi}{2}] \xrightarrow{\sin} [-1, 1]$$

is a one-to-one and onto function. The inverse function is called arcsin.

$$[-1, 1] \xrightarrow{\arcsin} [-\frac{\pi}{2}, \frac{\pi}{2}]$$

9.2 arcsin

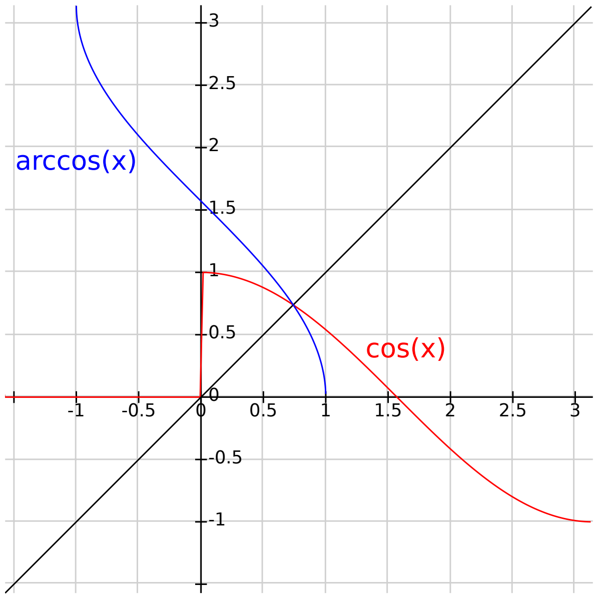
graphs of sin (red) and arcsin (blue)



9.3 arccos

For cos, we restrict the domain to the interval $[\,0\,,\,\pi\,]$. The inverse function is arccos: $[-1\,,\,1\,] \xrightarrow{\text{arccos}} [\,0\,,\,\pi\,]$

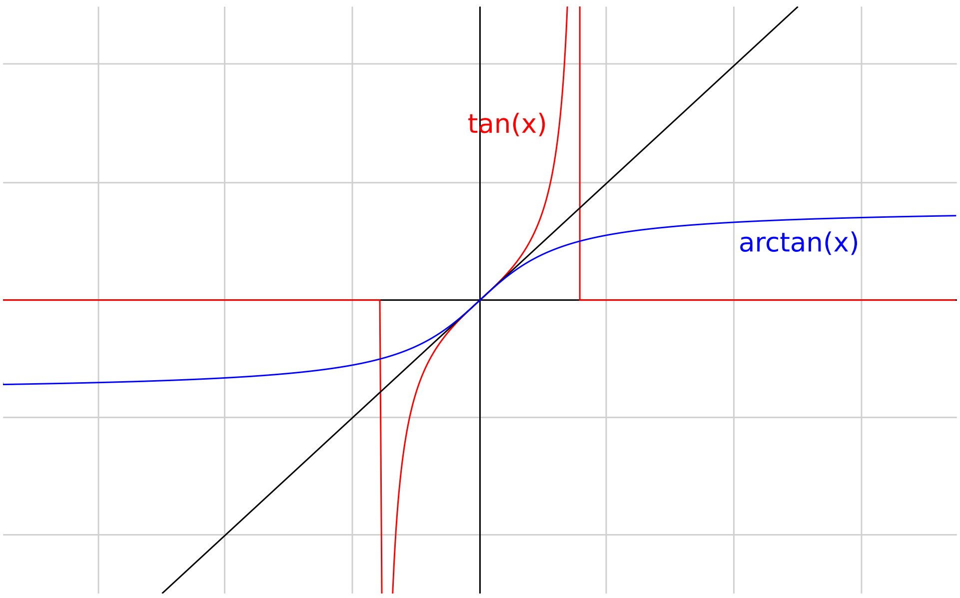
graphs of cos (red) and arccos (blue)



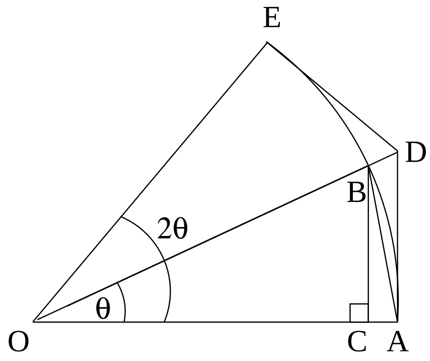
9.4 arctan

For tan, we restrict the domain to the open interval $(\,-\frac{\pi}{2}\,,\,\frac{\pi}{2}\,)$. The inverse function is arctan: $(-\infty\,,\,\infty\,) \xrightarrow{\text{arctan}} (\,-\frac{\pi}{2}\,,\,\frac{\pi}{2}\,)$

graphs of tan (red) and arctan (blue)



Consider a circular sector with center O and radius 1. Let A, B , E be points on the circular sector as shown.



Then:

$$\text{length}(BC) = \sin(\theta), \qquad \text{length}(\text{Arc(AB)}) = \theta, \qquad \text{and} \qquad \text{length}(AD) = \tan(\theta)\ .$$

14.1 1st inequalties and the limit $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

For $0 < \theta < \frac{\pi}{4}$, so 2θ is at most $\frac{\pi}{2}$ (90 degrees), we have:

$$\text{length}(BC) < \text{length}(AB) < \text{length}(\text{Arc(AB)}) \ , \ \text{so} \ \sin(\theta) < \text{length}(AB) < \theta$$

and

$$\text{length}(\text{Arc(AB)}) < \text{length}(AD) \ , \ \text{so} \ \theta < \tan(\theta).$$

In summary, we have:

$$\sin(\theta) < \theta < \tan(\theta)$$

We manipulate to:

$$\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1 \ , \ \text{valid for} \ \ 0 < \theta < \frac{\pi}{4}.$$

The three functions $\cos(\theta)$, $\dfrac{\sin(\theta)}{\theta}$, and 1 are all even, so the inequality is also true for $-\frac{\pi}{4} < \theta < 0$. In a picture:

The rule $\frac{\sin(\theta)}{\theta}$ does NOT allow input of $\theta = 0$. Zero is not in the domain. But, the function $\frac{\sin(\theta)}{\theta}$ is caught (for $\theta \neq 0$) between the two functions $\cos(\theta)$ and 1. We can apply the squeeze theorem to get

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \;=\; 1\,.$$

14.2 2nd limit involving the function $\frac{1-\cos(\theta)}{\theta}$

From the 1st picture we have:

$$\text{length}(AC)^2 \;+\; \text{length}(BC)^2 \;=\; \text{length}(AB)^2 \;\leq\; \text{length}(\text{Arc(AB)})^2\;, \quad \text{so}$$

$$(1-\cos(\theta))^2 \;+\; (\sin(\theta))^2 \;\leq\; \theta^2\;, \quad \text{then expand to get}$$

$$2\,(\,1 \;-\; \cos(\theta)\,) \;\leq\; \theta^2\;, \quad \text{and deduce}$$

$$0 \;\leq\; \frac{(\,1 \;-\; \cos(\theta)\,)}{\theta} \;\leq\; \frac{\theta}{2}\;, \quad \text{for } 0 \;<\theta\; < \frac{\pi}{4}$$

The (odd symmetry) rule $\frac{(\,1 \;-\; \cos(\theta)\,)}{\theta}$ does not allow input $\theta = 0$.

A picture of this rule (graph in red), with values of $\frac{(1 - \cos(\theta))}{\theta}$ caught between 0 (graph in yellow) and $\frac{\theta}{2}$ (graph in blue), is:

As before, we can deduce

$$\lim_{\theta \to 0} \frac{(1 - \cos(\theta))}{\theta} = 0 .$$

These two limits $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$, and $\lim_{\theta \to 0} \frac{(1 - \cos(\theta))}{\theta} = 0$ are very important. They are used later to compute the slope of tangent lines to the functions sin and cos.

15 Tangent slope of the functions sine and cosine

15.1 Tangent slope to the graph of sine at the point $(b, \sin(b))$.

The tangent slope at the graph point $(b, \sin(b))$ is the limit of the difference quotient:

$$\frac{\sin(b + h) - \sin(b)}{h} .$$

We use the formula for the sine of a sum to get:

$$\begin{aligned}
\sin(b + h) - \sin(b) &= \sin(b)\cos(h) + \cos(b)\sin(h) - \sin(b) \\
&= \sin(b)(\cos(h) - 1) + \cos(b)\sin(h)
\end{aligned}$$

$$\begin{aligned}
\lim_{h \to 0} \frac{\sin(b + h) - \sin(b)}{h} &= \lim_{h \to 0} \left(\sin(b) \frac{(\cos(h) - 1)}{h} + \cos(b) \frac{\sin(h)}{h} \right) \\
&= \sin(b) \lim_{h \to 0} \frac{(\cos(h) - 1)}{h} + \cos(b) \lim_{h \to 0} \frac{\sin(h)}{h} \\
&= \sin(b)\, 0 + \cos(b)\, 1 \\
&= \cos(b)
\end{aligned}$$

The tangent slope to the graph of sine at the point $(b, \sin(b))$ is $\cos(b)$.

15.2 Tangent slope to the graph of cosine at the point $(b, \cos(b))$.

The tangent slope at the graph point $(b, \cos(b))$ is the limit of the difference quotient:

$$\frac{\cos(\,b+h\,)\;-\;\cos(\,b\,)}{h}\,.$$

We use the formula for the cosine of a sum to get:

$$\begin{aligned}
\cos(\,b+h\,)\;-\;\cos(\,b\,) &= \cos(b)\,\cos(h)\;-\;\sin(b)\,\sin(h)\;-\;\cos(b)\\
&= \cos(b)\,(\,\cos(h)\;-\;1\,)\;-\;\sin(b)\,\sin(h)
\end{aligned}$$

$$\begin{aligned}
\lim_{h\to 0}\frac{\cos(\,b+h\,)\;-\;\cos(\,b\,)}{h} &= \lim_{h\to 0}\left(\cos(b)\,\frac{(\,\cos(h)\;-\;1\,)}{h}\;-\;\sin(b)\,\frac{\sin(h)}{h}\right)\\
&= \cos(b)\;\lim_{h\to 0}\frac{(\,\cos(h)\;-\;1\,)}{h}\;-\;\sin(b)\;\lim_{h\to 0}\frac{\sin(h)}{h}
\end{aligned}$$

Our earlier calculation using the squeeze theorem showed

$$\lim_{h\to 0}\frac{(\,\cos(h)\;-\;1\,)}{h}\;=\;0\quad\text{and}\quad\lim_{h\to 0}\frac{\sin(h)}{h}\;=\;1\,;$$

so,

$$\lim_{h\to 0}\frac{\cos(\,b+h\,)\;-\;\cos(\,b\,)}{h}\;=\;\cos(b)\;0\;-\;\sin(b)\;1\;=\;-\sin(b)$$

The tangent slope to the graph of cosine at the point $(b, \cos(b))$ is $-\sin(b)$.