Related rates.

Situation

- Two (or more quantities) are related to each other. For example quantities A and B are related to each other through a formula.
- Each quantity is a function of a 3rd variable. For example $A = A(t)$, and $B = B(t)$.
- **Goal**. Find the relationship between $A'(t)$, and $B'(t)$ using the relationship between $A(t)$ and $B(t)$.

Examples:

- Two planes are flying (directly) towards an airport. At a certain time:
  - Plane A is east of the airport and flying westward towards the airport with $A = 180$ miles, and $\dfrac{dA}{dt} = -120$ (miles/hr).
  - Plane B is south of the airport and flying northward with $B = 255$ miles, $\dfrac{dB}{dt} = -150$ (miles/hr).

  If L is the distance between A and B, determine $\dfrac{dL}{dt}$ at the given instant.

  We have $L(t) = \sqrt{(A(t))^2 + (B(t))^2}$. Apply the chain rule to get:

$$
\begin{aligned}
L'(t) &= \frac{1}{2}\Big((A(t))^2 + (B(t))^2 \Big)^{-\frac{1}{2}} \big((A(t))^2 + (B(t))^2 \big)' \\
&= \frac{1}{2}\Big((A(t))^2 + (B(t))^2 \Big)^{-\frac{1}{2}} \big(2\,A(t)\,A'(t) + 2\,B(t)\,B'(t) \big)
\end{aligned}
$$

  At the instant in question:

$$
\begin{aligned}
\frac{dL}{dt} &= \frac{1}{2}\frac{1}{\sqrt{(180)^2 + (255)^2}} \quad 2\big(180 \cdot (-120) + 255 \cdot (-150) \big) \ \text{ miles/hr} \\
&= -\frac{(\ 180 \cdot 120 + 255 \cdot 150\ )}{288.14\ldots} = -\,192.09\ldots \ \text{ miles/hr}
\end{aligned}
$$

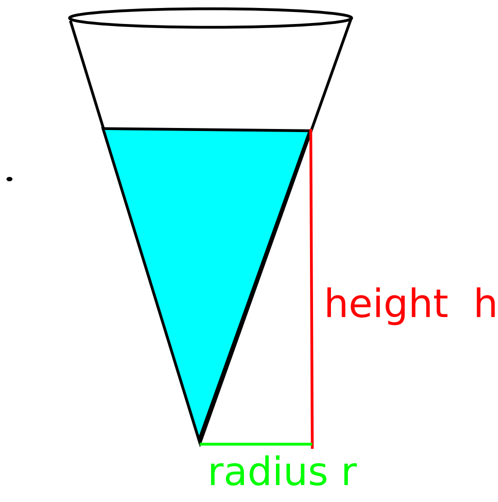
Summary of the solution process.

- Write down the relationship between the quantities.
- Differentiate the equation with respect to the 3rd variable.
- Substitute known values and solve for the desired quantity.

Examples:

- An inverted conical tank:
  - Has height 6 meters and radius 2 meters
  - The cone is leaking water at a rate of 10,000 cm^3/min.
  - Water is also being pumped in at a constant rate.
  - When the water level is 2 meters deep, the water level is rising at a rate of 20 cm/min.

  Find the rate at which water is being pumped into the cone.



Setup variables and relations: Let

$$\begin{aligned} h(t) &= \text{depth of water, and} \\ r(t) &= \text{radius at of top of water } = \frac{1}{3}h(t) \\ V(t) &= \text{volume } = \frac{1}{3}\text{area of base height } = \frac{1}{3}\pi(r(t))^2\, h(t) \\ &= \frac{1}{3}\;\pi\,(\frac{1}{3}h(t))^2\;\; h(t) \;=\; \frac{\pi}{27}h^3 \;\; (\text{writing } h \text{ for } h(t)) \end{aligned}$$

Differentiate and substitute known quantities:

$$\frac{dV}{dt} = \frac{\pi}{27}\,3\,h^2\,\frac{dh}{dt}$$

At the instant in question: $h = 200\text{cm}\,,\ \frac{dh}{dt} = 20\text{cm/min}.$ So,

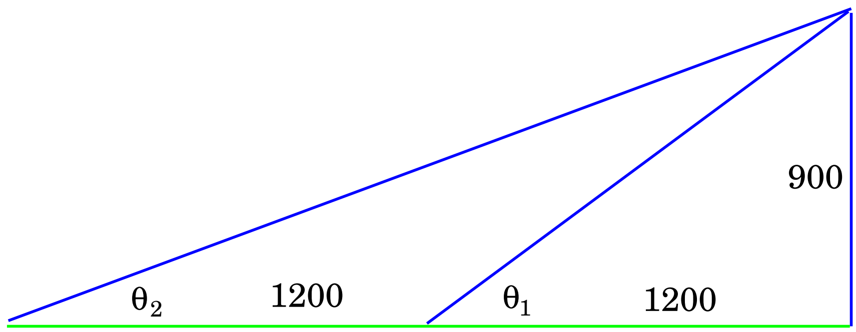
$$\left.\frac{dV}{dt}\right|_{h=200} = \frac{\pi}{27}\,3\,(200\text{ cm})^2\,(\,20\text{cm/min}\,) = \frac{800000\,\pi}{9}\,\text{cm}^3/\text{min} = 279,252.68\;\;\text{cm}^3/\text{min}$$

Since water is leaking from the bottom at a rate of 10,000 cm^3/min, the rate water is pumped in must be:

$$\left(\,10000 + \frac{800000\,\pi}{9}\,\right)\,\text{cm}^3/\text{min} = 289,252.68\;\;\text{cm}^3/\text{min}$$

- An agency wishes to film the launch of a rocket.
  - One camera is placed 1200 meters from the launch site, and a 2nd camera is placed (in line with the launch site and the first camera) at an additional distance of 1200 meters.
  - At launch, the rocket rises vertically, and at height 900 meters, it is measured that $\frac{d\Theta_1}{dt}$ is 10 degrees/sec (equal to $\frac{\pi}{180}10 = 0.1745\ldots$ radians/sec).

(i) Let $\Theta_1(t)$ be the angle of sight from camera 1 to the rocket, and $h(t)$ the height of the rocket. At the instant in question, determine Θ_1 and $\frac{dh}{dt}$.



We have:

$$\tan\,(\,\Theta_1(t)\,) = \frac{h(t)}{1200}\,, \quad \text{so} \quad \Theta_1(t) = \arctan\,(\,\frac{h(t)}{1200}\,)\,.$$

Take the derivative, to get

$$\frac{d\Theta_1}{dt} = \frac{d\arctan\left(\frac{h(t)}{1200}\right)}{dt} = \frac{1}{1+(\frac{h(t)}{1200})^2}\,\frac{d(\frac{h(t)}{1200})}{dt} = \frac{1}{1+(\frac{h(t)}{1200})^2}\,\frac{1}{1200}\,\frac{dh}{dt}$$

At the instant in question, when $h = 900$, we have:

$$\begin{aligned}
\Theta_1\Big|_{h=900} &= \arctan(\frac{900}{1200}) = 0.6435\ldots \text{ radians} \quad (36.8 \text{ degrees}) \\
\frac{dh}{dt}\Big|_{h=900} &= \left(1+(\frac{900}{1200})^2\right)\ 1200\ \frac{d\Theta_1}{dt}\Big|_{h=900} = \left(1+(\frac{900}{1200})^2\right)\ 1200\ (0.1745\ldots) \\
&= 327.24\ldots \quad \text{meters/sec}
\end{aligned}$$

(ii) Let $\Theta_2(t)$ be the angle of sight from camera 2 to the rocket. At the instant in question, determine Θ_2 and $\frac{d\Theta_2}{dt}$.

We have:

$$\tan(\Theta_2(t)) = \frac{h(t)}{2400}, \quad \text{so} \quad \Theta_2(t) = \arctan\left(\frac{h(t)}{2400}\right), \quad \text{and}$$

$$\begin{aligned}
\frac{d\Theta_2}{dt} &= \frac{d\arctan\left(\frac{h(t)}{2400}\right)}{dt} = \frac{1}{1+(\frac{h(t)}{2400})^2}\,\frac{d(\frac{h(t)}{2400})}{dt} \\
&= \frac{1}{1+(\frac{h(t)}{2400})^2}\,\frac{1}{2400}\,\frac{dh}{dt}
\end{aligned}$$

At the instant when $h = 900$, we have

$$\begin{aligned}
\Theta_2\Big|_{h=900} &= \arctan(\frac{900}{2400}) = 0.3587\ldots \text{ radians} \quad (20.5 \text{ degrees}) \\
\frac{d\Theta_2}{dt}\Big|_{h=900} &= \frac{1}{1+(\frac{900}{2400})^2}\,\frac{1}{2400}\ 327.24\ldots = 0.1195\ldots \text{ radians/sec} \quad (6.84 \text{ degrees/sec})
\end{aligned}$$