Applications of exponentials and logarithms.

We give some uses of exponentials and logarithms.

Exponentials and rate of change.

The exponential function $y = e^t$ has the remarkable property that its derivative is itself.

$$\frac{dy}{dt} = y .$$

This equation, relates the derivative function $\frac{dy}{dt}$ to the original functions y.
It is called a **differential equation**.

For the function $y = e^{k\,t}$, with k a constant, we have $\frac{dy}{dt} = e^{k\,t} \cdot k$; so the function y satisfies the differential equation:

$$\frac{dy}{dt} = k\,y .$$

This differential equation is extremely useful in expressing how certain quantities change in time.

Examples:

- Population growth. The growth of many organisms such as animals, vegetation, viruses, bacteria, etc, if provided with unlimited resources will grow (in time) at a rate proportional to their existing population. This can be written mathematically as the population function $P = P(t)$ satisfies the differential equation:

  $$P'(t) = k\,P(t) \quad \text{or in different notation} \quad \frac{dP}{dt} = k\,P ,$$

  with k a constant.

- Radioactive decay. Unstable radioactive elements have been observed to decay. Let $A(t)$ be the amount of the radioactive substance at time t. Then, it has been observed A satisfies the following:

  $$A'(t) = -\,k\,A(t) \quad \text{or in different notation} \quad \frac{dA}{dt} = -\,k\,A .$$

The derivative of the function $y = e^{B\,t}$, where B is a constant, is $\frac{dy}{dt} = e^{B\,t}\,B$; so it satisfies the differential equation

$$\frac{dy}{dt} = B\,y .$$

If we multiply $e^{B\,t}$ by a constant D to get $z = D\,e^{B\,t}$, then $\frac{dz}{dt} = D\,e^{B\,t}\,B = B\,D\,e^{B\,t} = B\,z$; so $z = D\,y$ also satisfies the same differential equation as y: the derivative function equals B times the function.

Fact. There are infinitely many solutions of the differential equation $\frac{dy}{dt} = e^{B\,t}\,B$; but they all have the form

$$y = D\,e^{B\,t}\,.$$

If the value $y(0)$ of y at $t = 0$ is known, then there is a unique solution given as

$$y(t) = y(0)\,e^{B\,t}\,.$$

Examples:

- Population growth. A bacteria culture:
  - · Initally contains 100 cells, and grows at a rate proportional to its size.
  - · Has grown to 420 cells after 1 hour.

(i) Determine the differential equation satisfied by the population function P.

We have $P(t) = P(0)e^{Bt} = 100e^{Bt}$ satisfies $P'(t) = BP(t)$. We need to find B.

$$420 = P(1) = 100\,e^{B1} = 100\,e^{B} \quad \text{so} \quad B = \ln(\frac{420}{100})$$

The function P therefore satisfies the differential equation

$$\frac{dP}{dt} = \ln(4.2)\,P\,, \quad \text{and} \quad P(t) = \ln(4.2)\,e^{\ln(4.2)\,t}\,.$$

(ii) Determine the number of bacteria and rate of growth at time $t = 3$ hours. We have

$$P\Big|_{t=3} = P(3) = 100\,e^{\ln(4.2)\,3} = 7409 \text{ cells (rounded from 7408.79)}$$

$$\frac{dP}{dt}\Big|_{t=3} = \ln(4.2)P\Big|_{t=3} = 10632.2\ldots \text{ cells/hr}$$

(iii) Determine when the population will reach 10,000 cells. We solve

$$10000 \;=\; 100\, e^{\ln(4.2)\, t}$$

to get

$$\begin{aligned} \ln(4.2)\, t &= \ln\Big(\frac{10000}{100}\Big) \\ t &= \frac{1}{\ln(4.2)} \ln\Big(\frac{10000}{100}\Big) \;=\; 3.20\ldots \text{ hours} \end{aligned}$$

- Radioactive decay. The differential equation for radioactive decay is

$$\frac{dA}{dt} \;=\; -kA\;.$$

In terms of the initial amount $A(0)$ at time $t = 0$, the solution is $A(t) \;=\; A(0)\, e^{-kt}$. An important observation is the following:

$$A(t + \frac{\ln(2)}{k}) \;=\; A(0)\, e^{-k\,(t + \frac{\ln(2)}{k})} \;=\; A(0)\, e^{-k\,t}\, e^{-k\frac{\ln(2)}{k}} \;=\; A(0)\, e^{-k\,t}\frac{1}{2} \;=\; \frac{1}{2}\, A(t)\;.$$

This means the amount at time $t + \frac{\ln(2)}{k}$ is half the amount at time t. The number $\frac{\ln(2)}{k}$ is called the half-life of the substance.

- Carbon dating objects using radioactive decay. The carbon isotope C_{14} is an unstable radioactive form of carbon. It has a half-life of 5730 years. This means, if $\frac{dA}{dt} \;=\; -kA$ is the differential equation satisfied by the amount $A(t)$ of C_{14} present, then

$$5730 \text{ years} \;=\; \frac{\ln(2)}{k} \quad \text{so} \quad k \;=\; \frac{\ln(2)}{5730} \quad \text{and} \quad A(t) \;=\; A(0)\, e^{-\frac{\ln(2)}{5730}\, t}\;.$$

If we have the remains of an 'ancient' organism, and it is known (by comparing the amount of stable C_{12}, to the amount of C_{14}), that 74% of C_{14} remains from the time when the organism was alive, estimate the age.

We have:

$$\begin{aligned} 0.74\,A(0) &= A(t) = A(0)\,e^{-\frac{\ln(2)}{5730}\,t} \\ -\frac{\ln(2)}{5730}\,t &= \ln(0.74) \\ t &= -\ln(0.74)\,\frac{5730}{\ln(2)} = 2500 \text{ years} \ \ (\text{rounded from } 2484.7\ldots) \end{aligned}$$

Continuous compound interest.

Funds deposited in a bank receive interest. The amount of interest is described in two parts:

- The interest rate paid per year.
- How often the interest is compounded.

Examples:

If a bank pays 5% interest per year, and the interest is compounded once a year, then

A starting amount A_0 after **one year** grows to $A_0(1+0.05)$.
A starting amount A_0 after N **years** grows to $A_0(1+0.05)^N$.

If the 5% interest is compounded p times (periods) per year, then the interest paid per period is $\frac{5\%}{p}$, and:

A starting amount A_0 after **one year** grows to $A_0\left(1+\frac{0.05}{p}\right)^p$.
A starting amount A_0 after N **years** grows to $A_0\left(1+\frac{0.05}{p}\right)^{pN}$.

Semiannual compound interest is when $p = 2$, **quarterly** compound interest is $p = 4$, and **daily** compound interest is $p = 365$.

Continuous compounding is when we let p go to infinity.

If r is the annual interest rate, it happens that:

$$\lim_{p\to\infty} \left(1 + \frac{r}{p}\right)^p \quad \text{exists.}$$

To see the limit exists and find its values, we set $y_p = (1 + \frac{r}{p})^p$. Then

$$\ln(y_p) = \ln\left(\left(1 + \frac{r}{p}\right)^p\right) = p\,\ln\left(1 + \frac{r}{p}\right) = \frac{\ln\left(1 + \frac{r}{p}\right)}{\frac{1}{p}}$$

We consider the function $f(x) = \ln(1 + rx)$. By the chain rule, the derivative is $f'(x) = \frac{1}{1+rx}\,r$, and so $f'(0) = r$). If we go back to the definition of derivative, this means:

$$r = f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\ln(1+rh) - \ln(1+0)}{h} = \lim_{h\to 0} \frac{\ln(1+rh)}{h}$$

If we set $h = \frac{1}{p}$, we see that as $p \to \infty$, that $h \to 0$, and so

$$\lim_{p\to\infty} \frac{\ln\left(1 + \frac{r}{p}\right)}{\frac{1}{p}} = \lim_{h\to 0} \frac{\ln(1+rh)}{h} = f'(0) = r\ .$$

So, as $p \to \infty$, we see $\ln(y_p)$ has limit r. We can take exponentials to get $y_p \to e^r$ as $p \to \infty$. So,

$$\lim_{p\to\infty} \left(1 + \frac{r}{p}\right)^p = e^r\ .$$

Summary:

A_0 **compounded continuously** at annual rate r grows to $A_0 e^r$ after one year.

Polynomial growth vs exponential growth.

Consider the two functions

$$f(x) \;=\; 2^x \qquad \text{and} \qquad g(x) \;=\; x^2\,.$$

If we increase the input from x to $x+1$, we see the ratios $\frac{f(x+1)}{f(x)}$ and $\frac{g(x+1)}{g(x)}$ are:

$$\frac{f(x+1)}{f(x)} \;=\; \frac{2^{x+1}}{2^x} \qquad \text{and} \qquad \frac{g(x+1)}{g(x)} \;=\; \frac{(x+1)^2}{x^2} \;=\; (1+\frac{1}{x})^2\,.$$

Increasing the input to 2^x by 1 results in a doubling of the output, while increasing the input to x^2 results in a multiplication of the output by 'only' $(1+\frac{1}{x})^2$. What we can conclude from this is that:

$$\lim_{x\to\infty} \frac{x^2}{2^x} \;=\; 0\,.$$

More generally, if $p(x)$ is ANY polynomial and b^x is any exponential with $b > 1$, then

$$\lim_{x\to\infty} \frac{p(x)}{b^x} \;=\; 0\,.$$

Exponential grwoth is always much much faster than polynomial growth.