Uses of differentials to estimate errors.

Recall the derivative notation $\frac{df}{dx}$ is the intuition:
the derivative tells us the **change in output** Δy (from $f(b)$)
in response to a **change of input** Δx at $x = b$.

$$\begin{aligned} \Delta y &= f(\,b + \Delta y\,) - f(b) \\ &\doteq f'(b)\,\Delta\,x \qquad \text{(approximately)} \end{aligned}$$

Examples.

- The radius of a sphere is measured to be $r = 84$ cm with a possible error of $\Delta r = \pm 0.5$ cm.
  - What is the surface area of the sphere?
    Recall $S = 4\pi r^2$, so $\frac{dS}{dr} = 8\pi r$. We have

    $$S = 4\,\pi\,84^2 = 88,668.2 \text{ cm}^2$$

    The uncertainty of $\Delta r = \pm 0.5$ cm in the measurement of the radius r, means there uncertainty in the area is

    $$\Delta S = \big(8\,\pi\,r\big)\,\Delta r = 8\,\pi\,84 \cdot (\pm 0.5) \text{ cm}^2 = 1,055.5 \text{ cm}^2$$

  - What is the volume of the sphere?
    Recall $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = 4\pi r^2$. We have

    $$V = \frac{4}{3}\,\pi\,84^3 = 2,482,712.6 \text{ cm}^3$$

    The uncertainty of $\Delta r = \pm 0.5$ cm in the measurement of the radius r, means the uncertainty in the volume is

    $$\Delta V = \big(4\,\pi\,r^2\big)\,\Delta r = 4\,\pi\,84^2 \cdot (\pm 0.5) \text{ cm}^3 = 4433.4 \text{ cm}^3$$

Uses of the tangent line.

We give two important uses of the tangent line to a graph:

- Accurate estimates of the function.
- A method (Newton's method) to determine where the graph of a function crosses the x-axis.

Accurate estimates of the function.

If $P = (b, f(b))$ is a graph point of a function f, and $m = f'(b)$ is the tangent slope at P, then the tangent line:

$$y \;=\; m\,(x - b) \;+\; f(b)$$

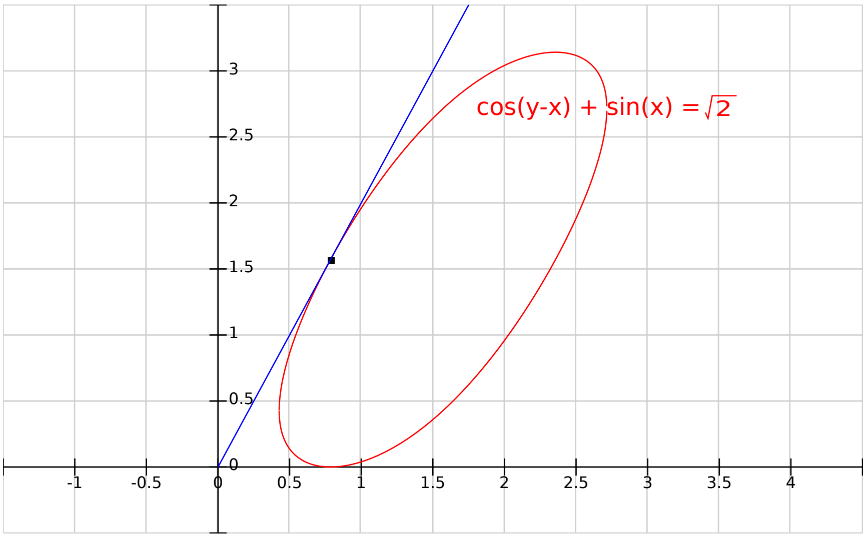
gives very accurate estimates of the function near the input $x = b$.

Example.

- Consider the function the implicitly defined function $y = y(x)$ which is defined by the equation:

  $$\cos\,(\,y \,-\, x\,) \,+\, \sin(\,x\,) \;=\; \sqrt{2}$$

  We looked at this in week 5. The point $P = (\frac{\pi}{4}, \frac{\pi}{2})$ lies on the graph, and the tangent slope at P is:



We can solve for y explicitly as $y = x + \arccos(\sqrt{2} - \sin(x))$.

We also use implicit differentiation to find the tangent slope at P:

$$\frac{d}{dx}\Big(\cos(y-x) + \sin(x)\Big) = \frac{d}{dx}(\sqrt{2}) = 0$$

$$-\sin(y-x)\Big(\frac{dy}{dx} - 1\Big) + \cos(x) = 0$$

$$\Big(\frac{dy}{dx} - 1\Big) + \frac{\cos(x)}{-\sin(y-x)} = 0$$

So,

$$\frac{dy}{dx} = 1 + \frac{\cos(x)}{\sin(y-x)}, \quad \frac{dy}{dx}\Big|_{(\frac{\pi}{4},\frac{\pi}{2})} = 1 + \frac{\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{2} - \frac{\pi}{4})} = 2$$

and the tangent line at $P = (\frac{\pi}{4}, \frac{\pi}{2})$ is

$$y = T(x) = 2\Big(x - \frac{\pi}{4}\Big) + \frac{\pi}{2}$$

The next table gives values of $y(x) = x + \arccos(\sqrt{2} - \sin(x))$ and $T(x)$ for x near $\frac{\pi}{4}$.

Δx	actual $= f(\frac{\pi}{4} + \Delta x)$	est. $= T(\frac{\pi}{4} + \Delta x)$	error (actual − est.)	relative error $\frac{\text{actual} - \text{est.}}{\Delta x}$
0.1	1.7616357	1.7707963	-0.009161	-0.091606
0.05	1.6684104	1.6707963	-0.002386	-0.047718
0.02	1.6104040	1.6107963	-0.000392	-0.019615
0.01	1.5906973	1.5907963	-0.000099	-0.009902
0.005	1.5807714	1.5807963	-0.000025	-0.004975
0.002	1.5747923	1.5747963	-0.000004	-0.001996

This example shows that the tangent line (at input $x = b$) can be used to estimate the values of a function for inputs near $x = b$. Not only will the error:

$$\text{error} \quad = \quad \big(\text{ true value at input } (b + \Delta x) \;\big) \; - \; \big(\text{ tangent line value at } (b + \Delta x) \;\big)$$

go to zero as $\Delta x \to 0$, but the relative error $\frac{\text{error}}{\Delta x}$ goes to zero too.

- Use the tangent line of the function $y = \sqrt{x}$ to give an estimate value for $\sqrt{4.01}$.

We know $\sqrt{4} = 2$, so $P = (4, 2)$ lies on the graph of the function. We have

$$\frac{dy}{dx} = \frac{1}{2}\, x^{-\frac{1}{2}} \,, \qquad \frac{dy}{dx}\Big|_{x=4} = \frac{1}{2}\, 4^{-\frac{1}{2}} = \frac{1}{4}$$

The tangent line at P is:

$$T(x) = \frac{1}{4}\,(\, x - 4 \,) + 2$$

The tangent line estimate for $\sqrt{4.01}$ is thus:

$$T(4.01) = \frac{1}{4}\,(\, 4.01 - 4 \,) + 2 = 2.0025 \,.$$

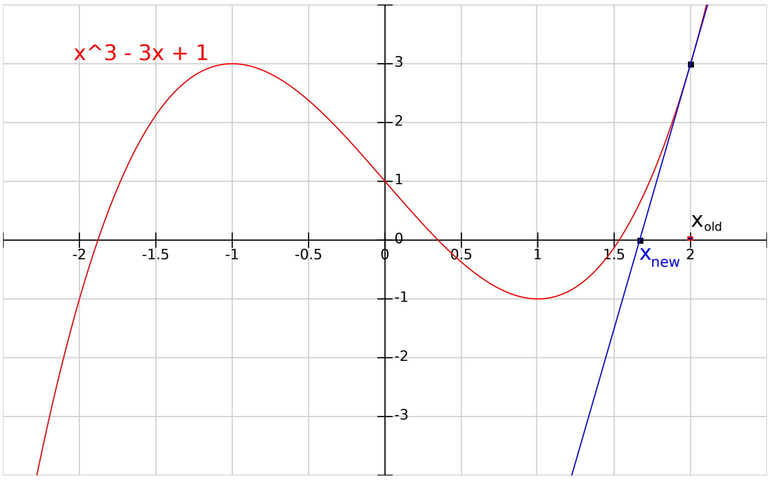
The actual value of $\sqrt{4.01}$ is $2.002498\ldots$

Newton's method.

This method was discovered by Sir Isaac Newton in the late 1600's to numerically solve for roots of equations.

Illustration of Newton's method.

The function $f(x) = x^3 - 3x + 1$ has 3 irrational roots. One of the roots is between 1.5 and 2.



Newton's method is to take a guess for the root – we take $x_{\text{old}} = 2$. If the guess is not a root, then follow the tangent line at $P = (x_{\text{old}}, f(x_{\text{old}}))$ to where it crosses the x-axis and call that point x_{new}.

Since $f'(x) = 3x^2 - 3$, the tangent slope at $P = (x_{\text{old}}, f(x_{\text{old}}))$ is $m_P = 3x_{\text{old}}^2 - 3$. Then,

$$m_P = \frac{f(x_{\text{old}})}{x_{\text{old}} - x_{\text{new}}}.$$

So,

$$x_{\text{new}} = x_{\text{old}} - \frac{f(x_{\text{old}})}{m_P} = x_{\text{old}} - \frac{x_{\text{old}}^3 - 3x_{\text{old}} + 1}{3x_{\text{old}}^2 - 3}$$

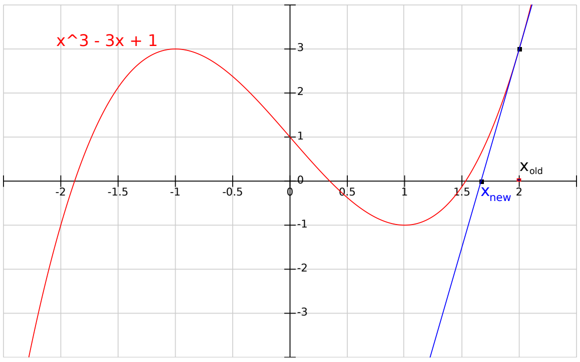
$$x_{\text{new}} = \frac{2x_{\text{old}}^3 - 1}{3x_{\text{old}}^2 - 3}$$

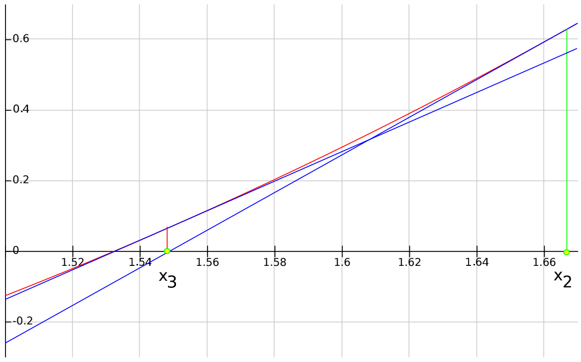
For our guess $x_0 = 2$, the new guess x_1 is then $x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 3} = \frac{15}{9} = 1.6666\ldots$.

Newton's method is to then take x_1 as our guess, and compute a new guess x_2 in the same fashion, so $x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 3} = 1.548611\ldots$.

x_{old}	x_{new}
2.000000	1.666667
1.666667	1.548611
1.548611	1.532390
1.532390	1.532088
1.532088	1.532088

There is a root at $1.532088\ldots$.





The equation $x^3 - 3x + 1 = 0$ has three roots. We can use Newton's method to determine the approximate have of the 3 roots. We take initial guesses of 2 and 0.5 and -2.5. We get:

x_0	x_1	x_2	x_3	x_4
2	2.666667	1.548661	1.532390	1.532088
0.5	0.347222	0.347296	0.347296	0.347296
-2.5	-2.047618	-1.897039	-1.879385	-1.879385

The three roots of $x^3 - 3x + 1 = 0$ are approximately 1.532088, 0.347296, and -1.879385.

Newton's method:

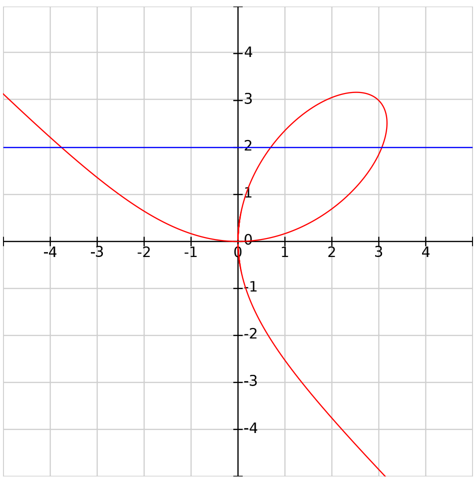
- Suppose a function f is a differentiable on an interval [a,b], and the graph crosses the x-axis at some point in the interior of the interval.
- Suppose x_0 is an initial guess of a root $f(x_{\text{root}}) = 0$ in the interval. Then if f is 'suitably nice', and the inital guess x_0 is close enough to the x_{root}, the sequence x_1, x_2, ... of roots of tangent lines given by

$$x_{\text{new}} = x_{\text{old}} - \frac{f(x_{\text{old}})}{f'(x_{\text{old}})}$$

will 'converge' to (have limit) x_{root}. We call the function $I(x) = x - \frac{f(x)}{f'(x)}$ the iteration function, and to the inital guess x_0, we have:

$$x_1 = I(x_0)\ ,\quad x_2 = I(x_1)\ ,\quad x_3 = I(x_2)\ ,\quad x_4 = I(x_3)\ ,\ \ldots$$

Example: Consider the set of points which satisfy $x^3 + y^3 - 6xy = 0$.



The line $y = 2$ intersects the graph in three points. Find numerical estimates of the three points.

When $y = 2$, the equation $x^3 + y^3 - 6xy = 0$ becomes

$$0 = x^3 - 12x + 8\ ;$$

so we need to find the roots of $f(x) = x^3 - 12x + 8 = 0$. We have $f'(x) = 3\,x^2 - 12$. If we use x_{old} as a guess for a root, then Newton's method says the next guess should be

$$x_{\text{next}} = x_{\text{old}} - \frac{f(x_{\text{old}})}{f'(x_{\text{old}})} = x_{\text{old}} - \frac{x_{\text{old}}^3 - 12x_{\text{old}} + 8}{3\,x_{\text{old}}^2 - 12}$$
$$= \frac{2x_{\text{old}}^3 - 8}{3\,x_{\text{old}}^2 - 12} = \frac{2}{3}\,\frac{x_{\text{old}}^3 - 4}{x_{\text{old}}^2 - 4}$$

x_0	x_1	x_2	x_3	x_4
3.2	3.073504	3.064226	3.064417	3.064417
0.9	0.683594	0.694569	0.694592	0.694592
-3.1	-4.015567	-3.780144	-3.758938	-3.758770

The three roots of $x^3 - 12x + 8 = 0$ are approximately 3.064417, 0.694592, and -3.758770.

Example where Newton's method fails.

We take g to be the odd continuous function

$$g(x) = \begin{cases} \sqrt{x} & 0 \le x \\ -\sqrt{-x} & x < 0 \end{cases}$$

which is differentiable for $x \neq 0$ and

$$g'(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & 0 < x \\ \dfrac{1}{2\sqrt{-x}} & x < 0 \end{cases}$$

If we take initial guess $x_{\text{old}} = a > 0$, the new guess is

$$x_{\text{new}} = a - \frac{g(a)}{g'(a)} = a - \frac{\sqrt{a}}{\frac{1}{2\sqrt{a}}} = a - 2\,a = -a\,(< 0)$$

Similarly if we take initial guess $x_{\text{old}} = -a < 0$, the new guess is

$$x_{\text{new}} = -a - \frac{g(-a)}{g'(-a)} = a - \frac{-\sqrt{a}}{\frac{1}{2\sqrt{a}}} = -\,a + 2\,a = a\,(> 0)$$

It follows that if we choose any $b \neq 0$, the sequence of guesses will just endlessly switch back and forth b, $-b$, b, $-b$, ...