The Mean Value Theorem.

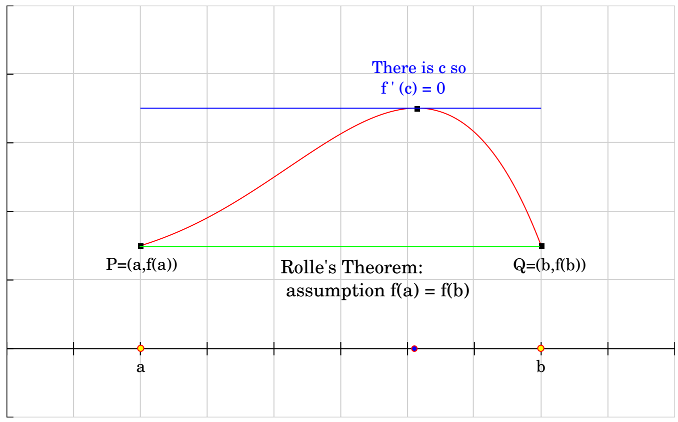
We have mentioned the intuition that if the derivative is positive, then the function is increasing, and if the derivative is negative the function is decreasing. This intuition was established in the 17th century by a fact called the Mean Value Theorem (MVT).

A special case is of the MVT is called Rolle's Theorem. We illustrate it and we will see how Rolle's Theorem and MVT actually follow from the global/local extreme value theorems.

Rolle's Theorem Suppose f is a function satisfying:

- Its domain is a closed interval $[a, b]$, and it is continuous on $[a, b]$.
- The derivative f' exists for all interior points.
- The function has the same value at the endpints a and b (so $f(a) = f(b)$).

Then there will be a point c in the interior with $f'(c) = 0$.



Explanation of Rolle's Theorem using the global/local extreme value theorems.

The hypothesis are:

1. Its domain is a closed interval $[a, b]$, and it is continuous on $[a, b]$.
2. The derivative f' exists for all interior points.
3. The function has the same value at the endpints a and b (so $f(a) = f(b)$).

- Hypothesis 1 is the hypothesis of the global extreme value theorem (EVT), so therefore the global EVT tells us the function f will that an absolute maximum as well as an absolute minimum.
- By Hypothesis 3 $f(a) = f(b)$, either the absolute maximum or the absolute minimum must occur at some interior point.
- If either the absolute maximum or the absolute minimum occurs at some interior point c, then Hypothesis 2 and the local extreme value theorem tells us $f'(c) = 0$.

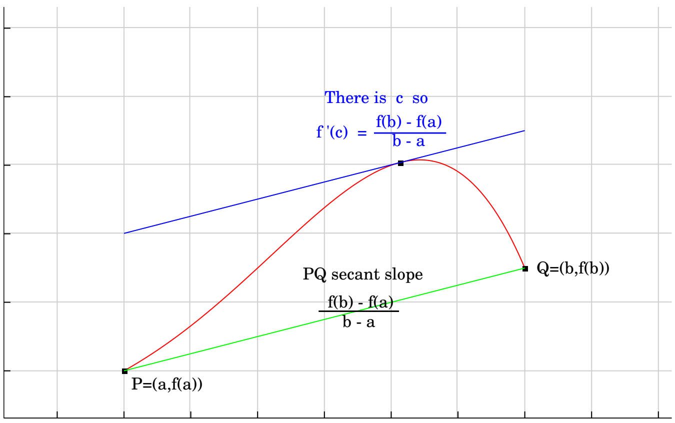
The Mean Value Theorem Suppose f is function satisfying:

- Its domain is a closed interval $[a, b]$, and it is continuous on $[a, b]$.
- The derivative f' exists for all interior points.

Then there will be a point c in the interior with

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

the slope of the secant line from $P = (a, f(a))$ to $Q = (b, f(b))$.



Note that Rolle's Theorem is 'merely' the Mean value Theorem when the PQ secant slope is zero. Conversely, if f satisfies the 2 hypotheses of the Mean Value Theorem, then the function

$$g(x) = f(x) - \Big(\frac{f(b) - f(a)}{b - a} (x - a) + f(a) \Big),$$

which is devised so that $g(a) = 0 = g(b)$, satisfies the 3 hypotheses of Rolle's Theorem. So there is a c so that $g'(c) = 0$. This means

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$

Therefore, Rolle's Theorem also implies the Mean Value Theorem.

Key uses of the Mean Value Theorem

The Mean Value Theorem (MVT) is used to justify three key facts:

- If the derivative of a function everywhere zero on an interval, then the function is constant.
- Justification that sign of derivative indicates whether function is increasing or decreasing.
- A useful rule for determining limits called L'Hôpital's rule.

Derivative zero everywhere on an interval $\Longrightarrow$ function constant

If a function has derivative zero everywhere on an interval, then the function is constant.

Reason. We show the values of f at any two point $x_1 < x_2$ are equal. This means the function is constant.

We apply the Mean Value Theorem to the interval $[x_1, x_2]$. Is says there is a interior point c so that:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) .$$

We use the hypothesis the derivative is zero everywhere to say $f'(c) = 0$, and therefore

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \quad \text{so} \quad f(x_2) = f(x_1) .$$

Sign of derivative

Suppose f is a continuous function on a closed interval $[a, b]$ and $f' > 0$ in the interior for the interval. Then f is **strictly increasing**, meaning:

$$x_1 < x_2 \implies f(x_1) < f(x_2) .$$

Justification. We apply the mean value theorem to the function f on the closed interval $[x_1, x_2]$. It states, there is a c in the interior (x_1, x_2) so that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) .$$

The hypothesis $f' > 0$ means $f'(c) > 0$, so $\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$, and therefore $f(x_2) - f(x_1) > 0$.