Examples analyzing functions by calculus.

Calculus was invented to analyze functions. Now that we have the tools of derivatives, we analyze some functions.

Example 1. The function

$$p(t) \;=\; \frac{1}{1 \;+\; a\,e^{-kt}} \quad (\text{constants } a,\; k \;>\; 0)\;,$$

with domain $[0, \infty)$ is used to model the spread of an illness (such as flu) through a population. The function p is the proportion of the population have already caught the illness. It satisfies the equation

$$\frac{dp}{dt} \;=\; k\,(\,1 \;-\; p\,)\;.$$

Initially the rate of inflection is approximately steady at k, but as a greater fraction of the population catches the illness, the rate of new infection slows down.

We use (i) limits, (ii) 1st derivative, and (iii) 2nd derivative to analyze the behavior of the function $p(t)$.

- Since $(1 \;+\; a\,e^{-kt}) \;>\; 1$, the value of p is always less than 1 which it should be if p measures the fraction of the population which has caught the illness.
- As $t \to \infty$, we have

$$\lim_{t\to\infty} e^{-kt} \;=\; 0 \quad \text{so} \quad \lim_{t\to\infty} \frac{1}{1 \;+\; a\,e^{-kt}} \;=\; 1\;.$$

  This corresponds to everyone eventually catching the illness. Graphically the horizontal line $y = 1$ will be an asympotote.
- The derivative $p'(t)$ equals

$$\begin{aligned} p'(t) \;&=\; \left(1 \,+\, a\,e^{-kt}\right)^{-1} \;=\; -\left(1 \,+\, a\,e^{-kt}\right)^{-2}(\,0 \,+\, a\,e^{-kt}\,(-k)\,) \\ &=\; \frac{a\,k\,e^{-kt}}{(\,1 \,+\, a\,e^{-kt}\,)^2} \end{aligned}$$

  Since $p'(t) \;>\; 0$, the function p is increasing. This corresponds to the number of people who have gotten the illness increasing over time. Graphically the function p increases towards the horizontal asymptote $y = 1$.
- Since the function is increasing, and $p(0) \;=\; \dfrac{1}{1 \,+\, a}$, and $\lim_{t\to\infty} p(t) \;=\; 1$, it's range of values is $[\dfrac{1}{1 \,+\, a}, 1)$. The number $p(0) \;=\; \dfrac{1}{1 \,+\, a}$ is the initial proportion of people whom are infected. Note that if $a > 1$, then $p(0) \;<\; \frac{1}{2}$ which means the initial proportion is less than 50%.

  That p is Increasing means the function p is one-to-one, and so has an inverse. To find the inverse we take:

$$p \;=\; \frac{1}{1 \;+\; a\,e^{-kt}}$$

  and solve for t in terms of p.

We have:

$$\begin{aligned} 1 + a e^{-kt} &= \frac{1}{p} \\ a e^{-kt} &= \frac{1}{p} - 1 = \frac{1-p}{p} \\ -kt &= \ln\left(\frac{1-p}{ap}\right) \\ t &= -\frac{1}{k} \ln\left(\frac{1-p}{ap}\right) = \frac{1}{k} \ln\left(\frac{ap}{1-p}\right) \quad \text{inverse function} \end{aligned}$$

We use the inverse function to find when p reaches 50%. We mention this needs $a \geq 1$ (otherwise we get a negative value for t). We see

$$t_{50\%} = \frac{1}{k} \ln\left(\frac{a\,0.50}{1 - 0.50}\right) = \frac{1}{k} \ln(a)\,.$$

· Intervals of concavity of up/down: They are determined by intervals where the 2nd derivative p'' is +/-.

$$\begin{aligned} p''(t) &= \left(a\,k\,(1 + a e^{-kt})^{-2} e^{-kt}\right)' = a\,k\,\left((1 + a e^{-kt})^{-2} e^{-kt}\right)' \\ &= a\,k\,\Big((-2)(1 + a e^{-kt})^{-3}(a e^{-kt}(-k))\, e^{-kt} + (1 + a e^{-kt})^{-2} e^{-kt}(-k)\Big) \\ &= a\,k^2\,(1 + a e^{-kt})^{-3} e^{-kt}\left(2a e^{-kt} - (1 + a e^{-kt})\right) \\ &= a\,k^2\,(1 + a e^{-kt})^{-3} e^{-kt}\left(a e^{-kt} - 1\right). \end{aligned}$$

So, p'' has the sign of $(a e^{-kt} - 1)$. We get

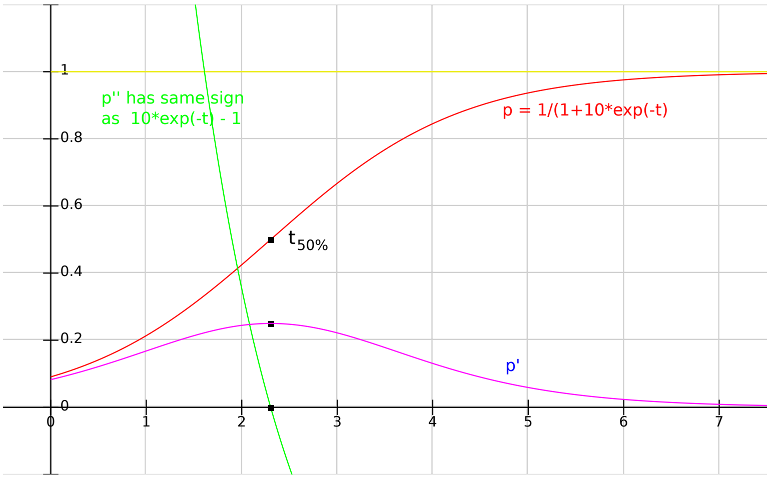
$$\text{sign of } p'' = \text{sign of } (a e^{-kt} - 1)$$

$$= \begin{cases} \text{always negative on } (0,\infty) \text{ when } 0 < a \leq 1. \\ \text{for } 1 < a, \text{ we have } \begin{cases} \text{positive for } 0 < t < \frac{1}{k}\ln(a) \\ \qquad (\text{Note. } t_{50\%} = \frac{1}{k}\ln(a)) \\ \text{positive for } \frac{1}{k}\ln(a) < t \end{cases} \end{cases}$$

The shape of the graph of p is:

- Always concave down when $0 < a \leq 1$.
- When $a > 1$, the curve is concave up in the interval $(0, t_{50\%})$, and then concave down in the interval $(t_{50\%}, \infty)$. The inflection point is $t_{50\%}$. The inflection point is where the p'' switches from positive to negative. This means p' has a local maximum at $t_{50\%}$. The interpretation is at $t_{50\%}$, the rate of spead of the rumor (illness) is at its greatest.

Graph when $a = 10$, $k = 1$.



Example 2. The function

$$f(x) = e^{-x^2/2} \quad \text{with domain } (-\infty, \infty)$$

has a 'bell curve' graph. This function has the name Gaussian density and is a important function used to study statistical patterns in very large populations – height/weight of people, exams scores, etc. Analyze this function for:

- behavior/limit at infinity
- local max/min
- intervals of concavity

Analysis:

Note the function f is even.

As $x \to \infty$, we have $-x^2/2 \to -\infty$, therfore,

$$\lim_{x\to\infty} e^{-x^2/2} = 0 \ .$$

Since f is even, the same is true as $x \to -\infty$. The x-axis is a horizontal asymptote of the function.

Next, we find the 1st derivative:

$$f'(x) \;=\; e^{-x^2/2}\,(-2x/2) \;=\; -x\,e^{-x^2/2}\,.$$

We deduce:

- The only critical point is $c = 0$. We also have $f' < 0$ when $x < 0$ and $f' > 0$ when $x > 0$. Therefore, f' flips sign from $+$ to $-$ at the critical point so it is a local maximum.
- The function is increasing on $(-\infty, 0]$, and decreasing on $[0, \infty)$. The local maximmum at input 0 must be a absolute maximum.
- Since $f(0) = 1$, the range of values of f is $(0, 1]$.

We determine concavity from the 2nd derivative:

$$\begin{aligned} f''(x) &= \big(-x\,e^{-x^2/2}\big)' \\ &= -\big(1\,e^{-x^2/2} + x\,(e^{-x^2/2}(-x))\big) \;=\; e^{-x^2/2}\,(x^2 - 1) \end{aligned}$$

Clearly,

$$\begin{aligned} f'' &> 0 \;\text{ concave up in the intervals } (-\infty, -1) \text{ and } (1, \infty) \\ f'' &< 0 \;\text{ concave up in the interval } (-1, 1) \\ x &= \pm 1 \;\text{ are inflection points} \end{aligned}$$

The inflection point distance from the center is called the standard deviation with this density. Here, it is 1.