# Anti-derivatives.

If $f$ is a function with domain an interval $\mathcal{I}$, an antiderivative is a function $F$ so that

$$F' \ = \ f$$

Examples.

- The functions $x^2$, $x^2{+}1$, $x^2{+}2$, and more generally $x^2{+}C$ ($C$ a constant) are antiderivatives to the function $f(x)$
- If $F$ is an antiderivative of $f$, then the function $G(x) \ = \ F(x) \ + \ C$ ($C$ a constant) is also an antiderivative.
- Conversely, suppose $F$, $G$ are both antiderivatives to a function $f$ on an interval $(a,b)$. Then

  $$\big(F \ - \ G\big)' \ = \ f \ - \ f \ = \ 0 \ \text{ zero function on the interval } (a,b)\,.$$

  But recall we used the Mean Value Theorem to say if the derivative of a function is zero on an interval $(a,b)$, then the function is constant. Therefore $(F \ - \ G)$ is a constant function. So,

  $$\begin{array}{c} F, \ G \ \text{ antiderivatves} \\ \text{for } f \text{ on an interval } (a,b) \end{array} \iff \begin{array}{c} (F \ - \ G) \ \text{ is a} \\ \text{constant function on } (a,b) \end{array}.$$

Examples

- The function $f(x) \ = \ \sin(2x)$ has domain $(-\infty,\infty)$. Find all antiderivatives of $f$ on the interval $(-\infty,\infty)$.

  We have $(\,\cos{(2x)}\,)' \ = \ (\,-\sin{(2x)}\,)\cdot 2$, so

  $$\big(-\frac{1}{2}\,\cos{(2x)}\big)' \ = \ -\frac{1}{2}\,(\,-\sin{(2x)}\,) \ = \ sin\,(2x)$$

  So, $F(x) \ = \ -\dfrac{1}{2}\,\cos{(2x)}$ is one anti-derivative to $f$ on the interval $(-\infty,\infty)$. All other anti-derivatives hav the form

  $$-\frac{1}{2}\,\cos{(2x)} \ + \ C \ \ (C \text{ a constant}).$$

- The function $f(x) \ = \ \ln(x)$ has domain $(0,\infty)$. Find all antiderivatives of $f$ on the interval $(0,\infty)$.

  We have $(x\,ln\,(x) \ - \ x)' \ = \ (\,1\cdot\ln{(x)} \ + \ x\dfrac{1}{x} \ - \ 1\,) \ = \ ln\,(x)$, so $(x\,ln\,(x) \ - \ x)$ is an anti-derivative. Any other anti-derivative has the form:

  $$(x\,\ln{(x)} \ - \ x) \ + \ C \ \ (C \text{ a constant}).$$

- Not all functions have anti-derivatives. The discontinuous function with domain $(-1, 1)$:

$$f(x) = \begin{cases} -1 & \text{for } -1 < x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } 0 < x \end{cases}$$

  does not have an anti-derivative on the entire interval $(-1, 1)$.

## Notation for the family of anti-derivatives

The anti-derivatives of a function $f$ (on an interval) form a family. The difference of any two members of the family is a constant function. Soon we will see that the Fundamental Theorem of Calculus connects anti-derivatives with things called integrals. Integrals of a function $f$ use the notation:

$$\int f(x)\,dx$$

to denote the family of anti-derivatives (when such anti-derivatives exist). The symbol, and the family of anti-derivative is called the **indefinite integral** of the function $f$.

Examples

- Find the indefinite integral $\int ( e^{2t} + 2\,t^{\frac{1}{2}} )\, dt$. This means find the family of anti-derivatives of the function $f(t) \ = \ e^{2t} \ + \ 2\,t^{\frac{1}{2}}$. We have

$$\Big(\frac{1}{2}\,e^{2t} \ + \ t^{\frac{3}{2}}\,\frac{4}{3}\Big)' \ = \ e^{2t} \ + \ 2\,t^{\frac{1}{2}} \ ;$$

so, the general anti-derivative of $f$ is

$$\int ( e^{2t} \ + \ 2\,t^{\frac{1}{2}} )\, dt \ = \ \frac{1}{2}\,e^{2t} \ + \ t^{\frac{3}{2}}\,\frac{4}{3} \ + \ C$$

- Find the indefinite integral $\int \frac{t+1}{t}\, dt \ = \ \int 1 \ + \ \frac{1}{t}\, dt$. We have

$$(\,t\,)' \ = \ 1 \quad \text{and} \quad (\,\ln(t)\,)' \ = \ \frac{1}{t} \ ;$$

so,

$$\int 1 \ + \ \frac{1}{t}\, dt \ = \ t \ + \ \ln(t) \ + \ C \ .$$

- Find the indefinite integral $\int (\sec(x))^2 \ - \ 1\, dx$. We have

$$(\,\tan(x)\,)' \ = \ (\sec(x))^2 \quad \text{and} \quad (\,x\,)' \ = \ 1 \ ;$$

so,

$$\int (\sec(x))^2 \ - \ 1\, dx \ = \ t \ + \ \tan(x) \ - \ x \ + \ C \ .$$

## Anti-derivative as a solution of a differential equation.

Recall, a differential equation is an equation for an unknown function $G$ which involves the derivatives $G'$ (and possibly higher derivatives). The equation that defines $G' = f$ is therefore a differential equation for the unknown function $G$. A solution to $G' = f$ is an anti-derivative of $f$.

Examples

- Let $p(t)$ be the position of an object on an axis, and suppose the speed $p'(t)$ equals $6t^2 + 4t - 10$. We have the differential equation

  $$p'(t) \;=\; 6t^2 \;+\; 4t \;-\; 10 \;\;,$$

  which is the assertion the function $p$ is an anti-derivative of $6t^2 + 4t - 10$. So,

  $$p(t) \;=\; 2t^3 \;+\; 2t^2 \;-\; 10t \;+\; C$$

  There is a family of solutions.

  **Initial value.** If we specify the value of $p$ at a specific time, say $p(0)$, there will be precisely one function in the family which satisfies the condition. The condition is called an **initial value condition**.

  Find the anti-derivative $p$ so that $p(0) = 0$. We have

  $$0 \;=\; p(0) \;=\; 2 \cdot 0^3 \;+\; 2 \cdot 0^2 \;-\; 10 \cdot 0 \;+\; C\;;$$

  so, $C = 0$, and $p(t) \;=\; 2t^3 \;+\; 2t^2 \;-\; 10t$.

- A car at speed $s_0$, and position $p(0) = 0$ breaks with constant deceleration of 5 meters/sec and produces skid marks of 60 meters before coming to a stop. Determine $s_0$, and how long $T$ it takes the car to stop.

  Let $p(t)$ be the position of the the car at time $t$, so

  $$\begin{aligned} p(0) \;&=\; 0\;, \quad \text{and} \quad p(T) \;=\; 60 \text{ (meters)} \\ p'(t) \;&=\; \text{speed}, \;\text{ and }\; p'(0) \;=\; s_0\;, \quad \text{and} \quad p'(T) \;=\; 0 \\ p''(t) \;&=\; \text{acceleration}, \;\text{ and }\; p''(t) \;=\; -\,5 \text{ (meters/sec)} \end{aligned}$$

  The speed function $p'(t)$ is an anti-derivative of the acceleration $p''(t)$, which is given as the function $-5$. Therefore,

  $$p'(t) \;=\; -5\,t \;+\; C_s \quad \text{where the constant } C_s \text{ needs to be determined}$$

  In turn, $p(t)$ is an anti-derivative of the speed $p'(t)$, so

  $$p(t) \;=\; -\frac{5}{2}t^2 \;+\; C_s\,t \;+\; C_p \quad \text{with the constant } C_s \text{ to be determined}$$

We use our initial conditions to get

$$\begin{aligned}
0 \;=\; p(0) \;&=\; -\frac{5}{2}\cdot 0^2 \;+\; C_s\cdot 0 \;+\; C_p \qquad \Longrightarrow \qquad C_p \;=\; 0 \\
0 \;=\; p'(T) \;&=\; -5\,T \;+\; C_s \qquad \Longrightarrow \qquad T \;=\; \frac{C_s}{5} \\
60 \;=\; p(T) \;&=\; -\frac{5}{2}\cdot T^2 \;+\; C_s\cdot T \qquad \Longrightarrow \qquad 60 \;=\; -\frac{5}{2}\cdot\frac{C_s^2}{5^2} \;+\; \frac{C_s^2}{5} \;=\; \frac{C_s^2}{10}\;.
\end{aligned}$$

So, $C_s^2 = 600 \;\;\Longrightarrow\;\; C_s = 10\sqrt{6} = 24.49$ meters/sec, and

$$\begin{aligned}
T \;&=\; \frac{C_s}{5} = 2\sqrt{6} = 4.89 \text{ seconds} \\
p'(t) \;&=\; -5t \;+\; 10\sqrt{6} \\
p'(0) \;&=\; -5\cdot 0 \;+\; 10\sqrt{6} \;=\; 24.49 \;\; \text{meters/sec}
\end{aligned}$$

The initial speed was $10\sqrt{6} \;=\; 24.49$ meters/sec.