L'Hôpital's rule for a limit.

L'Hôpital's rule for a limit allow one to sometimes find an indeterminant limit. An indeterminant limit is a limit of a ratio

$$\frac{f(x)}{g(x)}$$

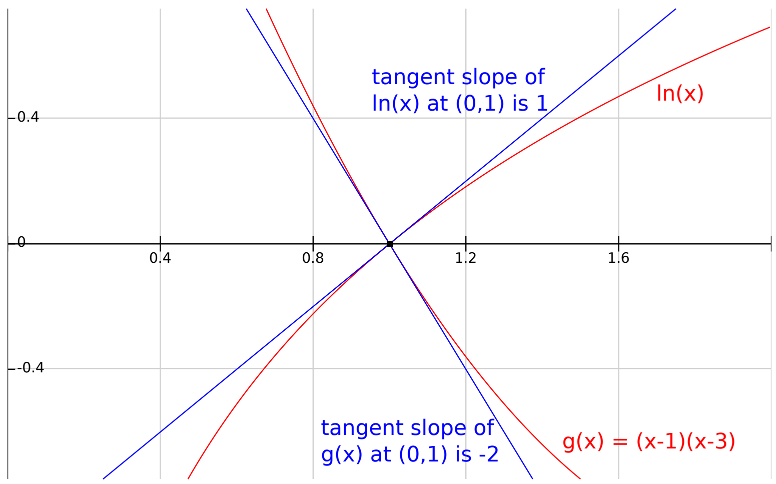
where both functions have either limit 0 or limit ∞.

Example. Consider

$$\lim_{x\to 1} \frac{\ln(x)}{(x-1)(x-3)} .$$

The two functions $\ln(x)$, and $g(x) = (x-1)(x-3)$ are both continuous on the interval $(0, \infty)$. Their limits as $x \to 1$, are $\ln(1) = 0$, and $g(1) = (1-1)(1-3) = 0$. Therefore, the limit of the ratio $\frac{f(x)}{g(x)}$ is indeterminant as $x \to 1$. The adjective indeterminant just refers to the fact that both the top and both function have limit 0.

The derivatives (tangent slopes) of $\ln(x)$ and $g(x)$ at input 1 are 1 and -2. These derivative values can be used to find the limit $\lim_{x\to 1} \frac{\ln(x)}{(x-1)(x-3)}$.



Since $\ln(1) = 0$ and $g(1) = 0$, we can write the ratio $\frac{\ln(x)}{g(x)}$ as:

$$\frac{\ln(x)}{g(x)} = \frac{\ln(x) - \ln(1)}{g(x) - g(1)} = \frac{\frac{\ln(x) - \ln(1)}{x-1}}{\frac{g(x) - g(1)}{x-1}}$$

Now, as $x \to 1$, we know:

$$\lim_{x\to 1} \frac{\ln(x) - \ln(1)}{x - 1} = (\ln)'(1) = \frac{1}{x}_{|x=1} = 1$$

$$\lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = (g)'(1) = (2x - 4)_{|x=1} = -2 .$$

By the quotient rule for limits:

$$\lim_{x \to 1} \frac{\frac{\ln(x) - \ln(1)}{x-1}}{\frac{g(x) - g(1)}{x-1}} = \frac{1}{-2} \qquad \text{so} \qquad \lim_{x \to 1} \frac{\ln(x)}{g(x)} = \frac{1}{-2} .$$

L'Hôpital's rule for a limit.

Assume f and g are two differentiable functions on an interval, and a is an interior point, and:

(i) The limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is indeterminant, that is,

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$.

(ii) $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$.

Then, $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ exists.

Examples.

- Suppose $a,\, b > 0$. Find $\lim_{x\to 0} \dfrac{a^x - b^x}{x}$.

  We check the individual limits.

  $$\lim_{x\to 0} (a^x - b^x) = 1 - 1 = 0 \qquad \text{and} \qquad \lim_{x\to 0} x = 0\ .$$

  so the original limit is indeterminant $\frac{0}{0}$.

  We take derivatives:

  $$(\, a^x - b^x\,)' = a^x \ln(a) - b^x \ln(b) \quad \Longrightarrow \quad \lim_{x\to 0} (\, a^x - b^x\,)' = \ln(a) - \ln(b)$$
  $$(\, x\,)' = 1 \quad \Longrightarrow \quad \lim_{x\to 0} (\, x\,)' = 1\ .$$

  So:

  $$\lim_{x\to 0} \frac{(a^x - b^x)'}{(x)'} = \frac{\ln(a) - \ln(b)}{1}\ .$$

  Therefore,

  $$\lim_{x\to 0} \frac{a^x - b^x}{x} = \ln(a) - \ln(b)$$

- Find $\lim_{x\to\frac{\pi}{2}} (\frac{\pi}{2} - x)\tan(x)$.

  As $x \to \frac{\pi}{2}$, we have $(\frac{\pi}{2} - x) \to 0$, and $\tan(x) \to \infty$, which is the indeterminant form $0 \cdot \infty$. We rewrite as:

  $$(\frac{\pi}{2} - x)\tan(x) = (\frac{\pi}{2} - x)\frac{\sin(x)}{\cos(x)} = \frac{(\frac{\pi}{2} - x)\sin(x)}{\cos(x)} = \frac{f(x)}{g(x)}$$

  and $\lim_{x\to\frac{\pi}{2}} f(x) = 0$ and $\lim_{x\to\frac{\pi}{2}} g(x) = 0$; so we get the standard indetermine form $\frac{0}{0}$. We check the 2nd hypotheses of L'Hôpital's rule.

  $$(\, f(x)\,)' = (0-1)\sin(x) + (\frac{\pi}{2} - x)\cos(x) \quad \Longrightarrow \quad \lim_{x\to\frac{\pi}{2}} (\, f(x)\,)' = -1$$
  $$(\, g(x)\,)' = -\sin(x) \quad \Longrightarrow \quad \lim_{x\to\frac{\pi}{2}} (\, g(x)\,)' = -1\ .$$

  So, $\lim_{x\to\frac{\pi}{2}} \dfrac{f'(x)}{g'(x)} = \frac{-1}{-1} = 1$, and therefore $(\frac{\pi}{2} - x)\tan(x) = \dfrac{f(x)}{g(x)}$ has limit 1, as $x \to \frac{\pi}{2}$.

Other forms of L'Hôpital's rule for a limit.

$\frac{\infty}{\infty}$ form.

Hypotheses

- $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$.
- $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)} = L$ exists.

Then, $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = L$ exists.

input approaching ∞ form.

Hypotheses

- $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to\infty} g(x) = 0$.
- $\lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)} = L$ exists.

Then, $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = L$ exists.

Caution.

One must properly check **both** hypotheses of L'Hôpital's rule.

Example. What is wrong with

$$\lim_{x\to\frac{\pi}{3}} \frac{\frac{1}{2} - \cos(x)}{\sin(x)} = \lim_{x\to\frac{\pi}{3}} \frac{\sin(x)}{\cos(x)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}\ ?$$

The calculation is wrong because $\lim\limits_{x\to\frac{\pi}{3}} (\frac{1}{2} - \cos(x)) = 0$, and $\lim\limits_{x\to\frac{\pi}{3}} (\sin(x)) = \frac{1}{2}$, so the ratio does not have an indeterminant $\frac{0}{0}$ form. By the 'regular' quotient rule for limits,

$$\lim_{x\to\frac{\pi}{3}} \frac{\frac{1}{2} - \cos(x)}{\sin(x)} = \frac{\lim\limits_{x\to\frac{\pi}{3}} (\frac{1}{2} - \cos(x))}{\lim\limits_{x\to\frac{\pi}{3}} (\sin(x))} = \frac{0}{\frac{1}{2}} = 0$$

Some idea why L'Hôpital's rule is true

The Mean Value Theorem, says if a function f is continuous on the interval $[a, b]$, and differentiable on the interval (a, b), then the secant slope

$$\frac{f(b) - f(a)}{b - a}$$

will be the value derivative at some interior point c. So there is a interior c with $f'(c)$ equal to the secant slope. There is a two function version of the Mean Value Theorem which says if f and g are two functions continuous on $[a, b]$, and differentiable on (a, b), then there is an interior point c so:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Suppose $\lim_{x \to d} \frac{f(x)}{g(x)} = \frac{0}{0}$ is indeterminant. This means $f(d) = 0$ and $g(d) = 0$; therefore, applying the two function Mean Value Theorem to f and g on the interval $[d, x]$

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(d)}{g(x) - g(d)} = \frac{f(c)}{g(c)} \quad \text{for some } c \in (d, x) \ .$$

As $x \to d$, the interior point c is squeezed between d and x and so $c \to d$. But, then $\frac{g'(c)}{f'(c)} \to L$, so $\frac{g(x)}{f(x)} \to L$ too.