## The Fundamental Theorems of Calculus.

We saw the example of how Riemann sums and limits can be used to find the area $\int_0^b x^2\,dx$ equals $\frac{b^3}{3}$. In the 17th century it was realized the process of integration is the reverse of differentiation. This discovery, now called called the fundamental theorems of calculus revolutionized what people could calculate. The word calculus means a way to calculate.

If a function $f$ with domain $[a,b]$ is integrable, then it is integrable on any subinterval $[c,d]$ inside $[a,b]$. Set

$$A(s) \;=\; \int_a^s f(t)\,dt$$

The area function $A(s)$ has values:

$$A(a) \;=\; \int_a^a f(t)\,dt \;=\; 0 \quad , \quad \text{and} \quad A(b) \;=\; \int_a^b f(t) \;.$$

# Fundamental Theorem of Calculus I

Suppose a function $f$, with domain $[a,b]$, is **continuous**. Take $A$ to be the 'area' function:

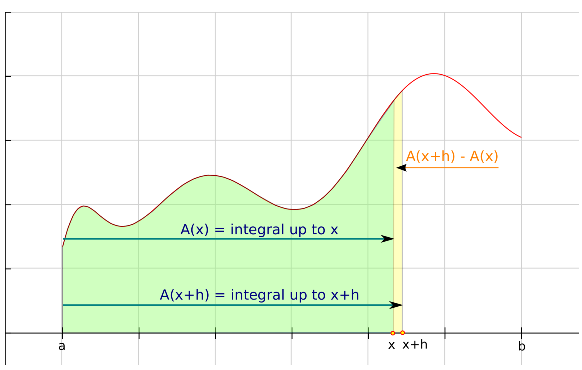
$$A(s) \;=\; \int_a^s f(t)\,dt \;.$$

Then, the function $A$ is continuous on $[a,b]$, and differentiable in the interior, and $A$ is an **anti-derivative** of $f$ on the interval $(a,b)$.

We explain the idea why the Fundamental Theorem of Calculus I is true. We go back to the definition for the derivative. For $x$ in the interior we need to show the difference quotient

$$\frac{A(x+h) \;-\; A(x)}{h}$$

has limit (as $h \to 0$) equal to $f(x)$.



By the picture, the difference $A(x+h) \; - \; A(x)$ is the definite integral (area) over the interval $[x, (x+h)]$.

$$\begin{aligned} A(x+h) \; - \; A(x) \; &= \; \int_a^{x+h} f(t)\, dt \; - \; \int_a^{x+h} f(t)\, dt \\ &= \; \int_x^{x+h} f(t)\, dt \;\; , \quad \text{and} \\ \frac{A(x+h) \; - \; A(x)}{h} \; &= \; \frac{1}{h} \int_x^{x+h} f(t)\, dt \end{aligned}$$

The expression $\frac{1}{h} \int_x^{x+h} f(t)\, dt$ is the key. Since $f$ is assumed to be continuous if we take $h$ sufficiently small the values of $f$ on the subinterval $[x, x+h]$ will be very close to $f(x)$, and so the definite integral $\int_x^{x+h} f(t)\, dt$ will be approximately $h \cdot f(x)$ and the difference quotient will have approximately value $f'(x)$. As we let $h \to 0$, we get

$$\lim_{h \to 0} \frac{A(x+h) \; - \; A(x)}{h} \; = \; \lim_{h \to 0} \; \frac{1}{h} \int_x^{x+h} f(t)\, dt \; = \; f(x)\; .$$

So, $A$ is an anti-derivative of $f$.

Examples.

- Find the area function $A(x) \; = \; \displaystyle\int_1^x \frac{1}{\sqrt{t}}\, dt$, and use it to determine the definite integral $\displaystyle\int_1^8 \frac{2}{\sqrt{t}}\, dt$.

  The Fundamental Theorem of Calculus I (FTC I) says the area function is an anti-derivative of the function $\frac{1}{\sqrt{x}}$. We find the family of anti-derivatives of $\frac{1}{\sqrt{x}} \; = \; t^{-\frac{1}{2}}$.

  $$\begin{aligned} (\, t^{\frac{1}{2}}\, )' \; = \; \frac{1}{2}\, t^{-\frac{1}{2}} \quad &\Longrightarrow \quad (\, 2\, t^{\frac{1}{2}}\, )' \; = \; t^{-\frac{1}{2}} \\ &\Longrightarrow \quad A(t) \; = \; 2\, t^{\frac{1}{2}} \; + \; C \quad \text{for some constant } C \text{ to be determined.} \end{aligned}$$

  To determine the constant $C$, we use the initial condition $A(1) \; = \; 0$:

  $$0 \; = \; A(1) \; = \; 2 \cdot 1 \; + \; C \quad \Longrightarrow C \; = \; -2\; , \text{ so } \; A(t) \; = \; 2\, t^{\frac{1}{2}} \; - \; 2\; .$$

  Then,

  $$\int_1^8 \frac{1}{\sqrt{t}}\, dt \; = \; A(8) \; = \; 2\sqrt{8} \; - \; 2 \; = \; 4\, \sqrt{2} \; - \; 2\; .$$

- Calculate the definite integral $\int_0^{\pi} \sin(t)\ dt$. We calculate the area function $A(x) = \int_0^x \sin(t)\ dt$ satisfying the initial condition $A(0) = 0$. Byt FTC I, it is an anti-derivative. But

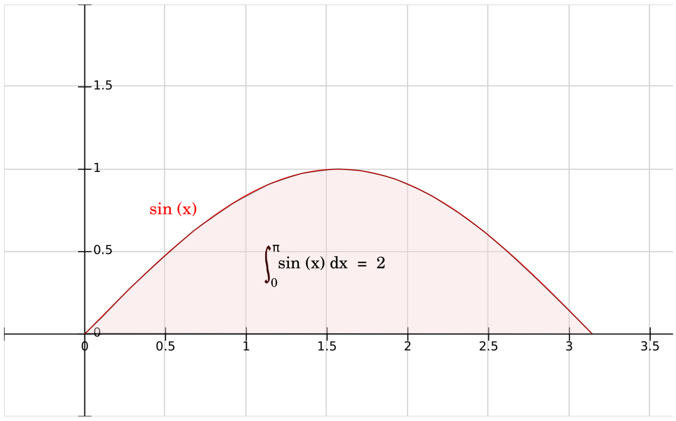
$$-\cos(x)\ +\ C \quad \text{with } C \text{ a constant is the general anit-derivatve}$$

So,

$0\ =\ A(0)\ =\ -\cos(0)\ +\ C\ = -1\ +\ C \quad \Longrightarrow\ C\ =\ 1\ ,\ \text{and}\ \ A(t)\ =\ -\cos(x)\ +\ 1\ .$

Then,

$$\int_0^{\pi} \sin(t)\ dt\ =\ A(\pi)\ = -\cos(\pi)\ +\ 1\ =\ -(-1)\ +\ 1\ =\ 2\ .$$




- Write the function $f(x)\ =\ \int_0^{\sin(x)} \sqrt{1+t^2}\ dt$ as a composite function and use the chain rule to compute $f'(x)$.

We have $f(x)\ =\ F(\sin(x)\ ,)$ where $F(y)\ =\ \int_0^y \sqrt{1+t^2}\ dt$. By FTC I, the derivative $F'(y)$ equals $\sqrt{1+y^2}$; so

$$f'(x)\ =\ F'(\sin(x))\ (\sin(x))'\ =\ \sqrt{1\ +\ (\sin(x))^2}\ \cos(x)\ .$$

# Fundamental Theorem of Calculus II

Suppose a function $f$, with domain $[a,b]$, is a **continuous**, and $F$ is **any** anti-derivative of $f$, then

$$\int_a^b f(t)\,dt \;=\; F(b) \;-\; F(a)\ .$$

Explanation. We set $A(x) = \int_a^x f(t)\,dt$, so $A(b)$ is the value of the definite integral. We have (i) $A$ is an anti-derivative of $f$ and so $A(x) = F(x) + C$ for some constant $C$, and we have the initial condition $A(a) = 0$.

$$0 \;=\; A(a) \;=\; F(a) \;+\; C \quad\Longrightarrow\quad C \;=\; -F(a)\ .$$

Then, $A(x) = F(x) - F(a)$, and so

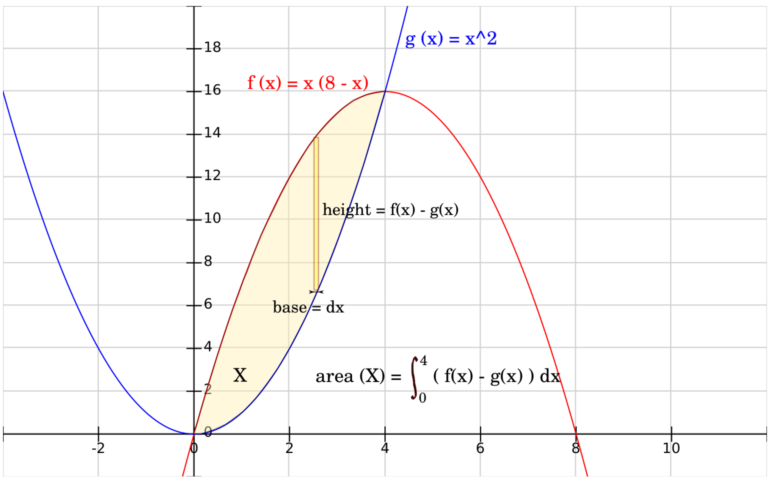
$$\int_a^b f(t)\,dt \;=\; A(b) \;=\; F(b) \;-\; F(a)\ .$$

Notation: We use the notation

$$F(x)\,\Big|_a^b \quad \text{to denote the difference} \quad F(b) \;-\; F(a)$$

Examples.

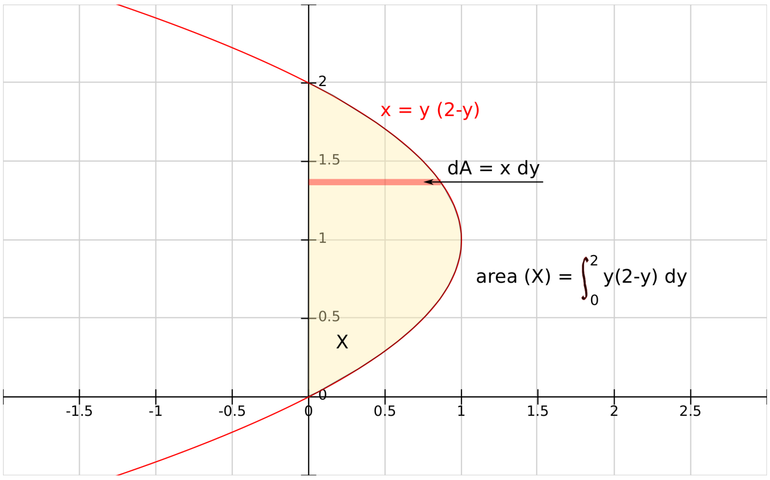
- The graphs of the two functions $f(x) = x\,(8-x)$ and $g(x) = x^2$ intersect at the two points $P = (0,0)$ and $Q = (4,16)$ and bound a region $X$. Find the area of $X$.




$$\text{area}\,(X) \;=\; \int_0^4 f(x) \;-\; g(x)\,dx \;=\; \int_0^4 8\,x \;-\; x^2 \;-\; x^2\,dx$$
$$=\; \left(\,4\,x^2 \;-\; \frac{2}{3}x^3\,\right)\Big|_0^4 \;=\; (\,4\cdot 4^2 \;-\; \frac{2}{3}4^3\,) \;-\; 0 \;=\; \frac{64}{3}\ .$$

- Let $X$ be the region bounded by the $y$-axis and the curve $x = 2y - y^2$. Find the area of $X$. We do so in two ways:

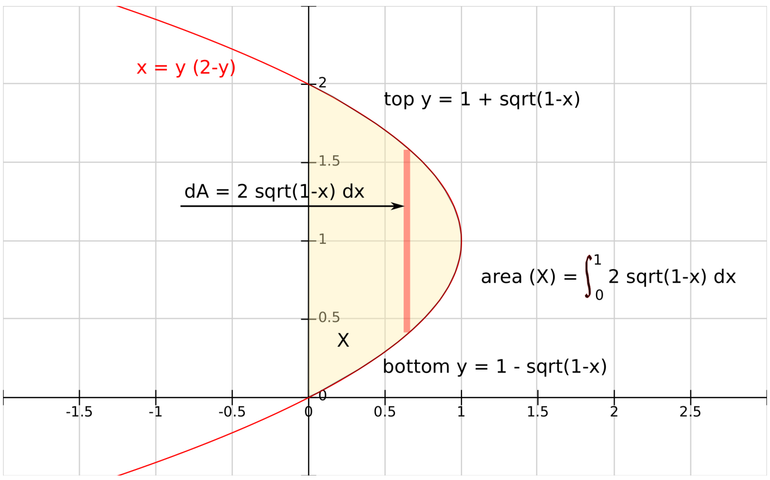
Solution 1, uses 'horizontal' rectangles.




The area of the infinitesimal rectangle of length $x$ and width $dx$ is $dA = x\,dy$. We get

$$\text{area}\,(X) \;=\; \int_0^2 x\,dy \;=\; \int_0^2 (y\cdot(2-y))dy \;-\; \int_0^2 2\,y \;-\; y^2\;dy$$
$$=\left(\,y^2 \;-\; \frac{y^3}{3}\,\right)\Big|_0^2 \;=\; \left(\,2^2 \;-\; \frac{8}{3}\,\right) \;=\; \frac{4}{3}$$

Solution 2, uses 'vertical' rectangles.




The infinitesimal 'vertical' rectangle has height $2\sqrt{1-x}$ and width $dx$, so its area is $dA = 2\sqrt{1-x}\,dx$

$$\text{area}\,(X) \;=\; \int_0^1 (1 \;+\; \sqrt{1-x}) \;-\; (1 \;-\; \sqrt{1-x})\;dx$$
$$=\; \int_0^1 2\,\sqrt{1-x}\;dx \;=\; -\frac{4}{3}\,(1-x)^{\frac{3}{2}}\,\Big|_0^1 \;=\; \frac{4}{3}$$

- Caution in using FTC II. The hypothesis of FTC II is that $f$ is continuous. One must be sure to verify the hypothesis. The following integral of the positive function $f(x) = \frac{1}{x^2}$ is invalid:

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -\left(\frac{1}{1} - \frac{1}{-1}\right) = -2 \ .$$

The function $\frac{1}{x^2}$ is not continuous on the interval $[-1,1]$. $\lim_{x\to 0} \frac{1}{x^2}$ does not exists.