Riemann sums and existence of the definite integral.

In the calculation of the area of the region X bounded by the graph of $g(x) = x^2$, the x-axis and $0 \leq x \leq b$, two sums appeared:

$$\Big(\sum_{k=1}^{n} (k-1)^2\Big)\, \frac{b^3}{n^3} \quad \leq \quad \text{area}(X) \quad \leq \quad \Big(\sum_{k=1}^{n} k^2\Big)\, \frac{b^3}{n^3}$$

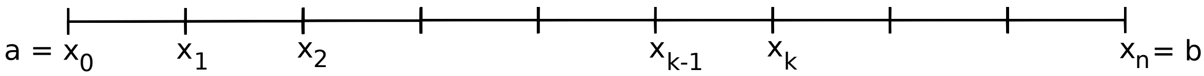
These sums are examples of what are called **Riemann sums**.

Equal length subintervals Riemann sum.

Suppose f is a function with domain $a \leq x \leq b$. We create a (equal length subintervals Riemann sum as follow:

- Take a positive integer and divide the interval $[a, b]$ in n equal length subintervals of length $\Delta x_n = \left(\frac{b-a}{n}\right)$. Clearly the endpoints of the various subintervals are:

$$x_0 = a \ , \quad \cdots \quad , \quad x_k = x_0 + k\,\Delta x_n \ , \quad \cdots \quad , \quad x_n = x_0 + n\,\Delta x_n = b \ .$$

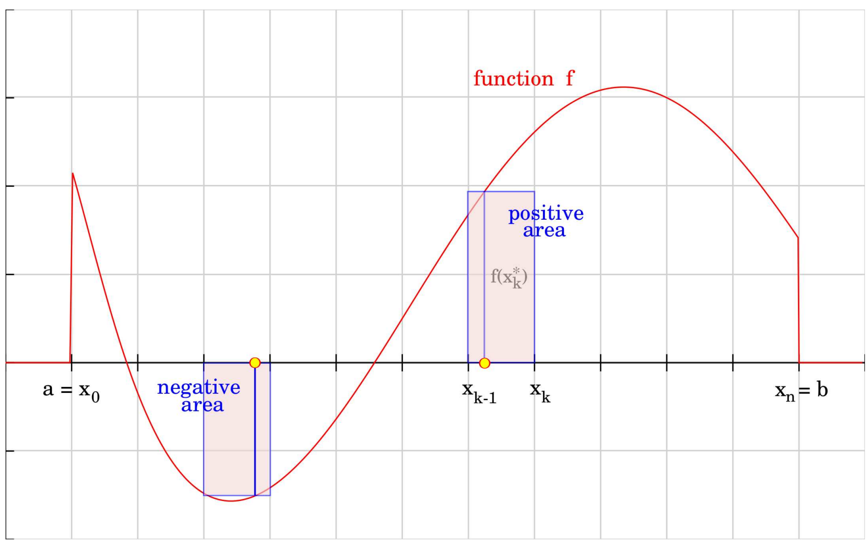


- For each subinterval $I_k = [x_{k-1}, x_k]$ we choose a point x_k^* in I_k.

Image a rectangle with base the interval I_k and height $f(x_k^*)$. The sum

$$\sum_{k=1}^{n} f(x_k^*)\, \Delta x_n \ , \quad (\text{where } \Delta x_n = \Big(\frac{b-a}{n}\Big)\ )$$

is called a Riemann sum.



We are free to choose the points x_k^* to be any point we want in the interval J_k, so there are many Riemann sums.

Examples

- **Left endpoint Riemann sum.** Here, we always take the point x_k^* to be the left endpoint $x_{k-1} = a + (k-1)\Delta x_n$ in the subinterval $I_k = [x_{k-1}, x_k]$. The Riemann sum is

$$\sum_{k=1}^{n} f(x_{k-1})\,\Delta x_n = \sum_{k=1}^{n} f(\,a + (k-1)\,\Delta x_n)\,\Delta x_n \quad , \quad (\text{where } \Delta x_n = (\frac{b-a}{n})\,)$$

- **Right endpoint Riemann sum.** Here, we take x_k^* to always be the right endpoint of I_k. The Riemann sum is

$$\sum_{k=1}^{n} f(x_k)\,\Delta x_n = \sum_{k=1}^{n} f(\,a + k\,\Delta x_n)\,\Delta x_n$$

- **Midpoint Riemann sum.** We take x_k^* to always be the midpoint $\frac{x_{k-1}+x_k}{2}$ of I_k. The midpoint is $a + (k - \frac{1}{2})\Delta x_n)$, and the Riemann sum is

$$\sum_{k=1}^{n} f(\frac{x_{k-1}+x_k}{2})\,\Delta x_n = \sum_{k=1}^{n} f(\,a + (k - \frac{1}{2})\,\Delta x_n)\,\Delta x_n$$

Nice property. Suppose the function f has domain $[a, b]$, and on any subinterval $[c, d]$, the function achieves a minimum and a maximum value.

Examples of functions which satisfy the nice property are:

- Any continuous function.
- An 'increasing' function – meaning $x_1 < x_2 \implies f(x_1) \le f(x_2)$.
- A 'decreasing' function – meaning $x_1 < x_2 \implies f(x_1) \ge f(x_2)$.

If the function f satisfies the nice property, for each positive integer n, we can create two Riemann sums by taking $x_k^{\min}$ to be an input which provides the minimum value of f on I_k, and $x_k^{\max}$ to be an input which provides the maximum value of f on I_k. We call the sums:

$$\sum_{k=1}^{n} f(\,x_k^{\min}\,)\,\Delta x_n \quad \text{an } \textbf{under-estimate Riemann sum}$$

$$\sum_{k=1}^{n} f(\,x_k^{\max}\,)\,\Delta x_n \quad \text{an } \textbf{over-estimate Riemann sum}$$

For any other x_k^* in I_k, we have $f(\, x_k^{\min}\,) \le f(\, x_k^*\,) \le f(\, x_k^{\max}\,)$, so

$$\sum_{k=1}^{n} f(\, x_k^{\min}\,)\, \Delta x_n \;\le\; \sum_{k=1}^{n} f(\, x_k^*\,)\, \Delta x_n \;\le \sum_{k=1}^{n} f(\, x_k^{\max}\,)\, \Delta x_n$$

which says any Riemann sum for the partition into n equal length subintervals of length $\Delta x_n = (\frac{b-a}{n})$ is between the under and the over estimate Riemann sums.

A function f satisfying the nice property is **integrable** if the under and over estimate Riemann sums have **equal** limits as $n \to \infty$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\, x_k^{\min}\,)\, \Delta x_n \qquad \text{equals} \qquad \lim_{n\to\infty} \sum_{k=1}^{n} f(\, x_k^{\max}\,)\, \Delta x_n\ .$$

The common limit value is called **the definite integral of the function**, and denoted as:

$$\int_a^b f(x)\, dx$$

Note. For the example $g(x) = x^2$ with domain $[0, b]$, the left endpoints of the intervals I_k's give the under-estimate Riemann sum:

$$\sum_{k=1}^{n} f(\, x_k^{\min}\,)\, \Delta x_n \;=\; \sum_{\ell=0}^{n-1} \ell^2 \; (\frac{b^3}{n^3}) \;\to\; \frac{b^3}{3} \quad \text{as } n \to \infty$$

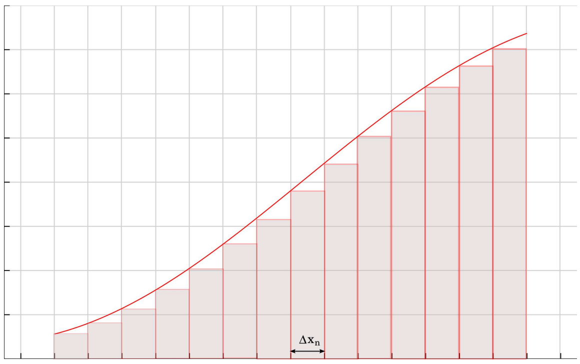
The right endpoints of the intervals I_k give the over-estimate Riemann sum:

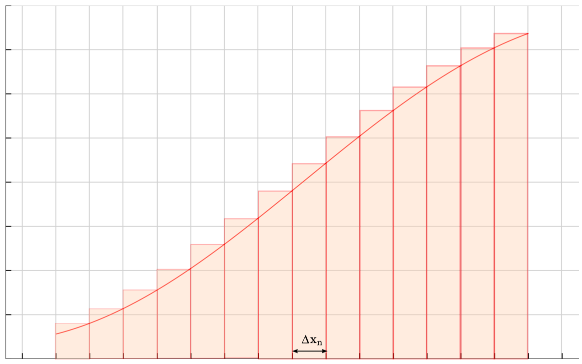
$$\sum_{k=1}^{n} f(\, x_k^{\max}\,)\, \Delta x_n \;=\; \sum_{k=1}^{n} k^2 \; (\frac{b^3}{n^3}) \;\to\; \frac{b^3}{3} \quad \text{as } n \to \infty$$

Existence of definite integral for increasing functions

Suppose a function f with domain $[a, b]$ is increasing (or decreasing) on the interval. Then, the under and over estimate Riemann sums have limits which are the same. So, the function is integrable.

These pictures give the reason why the assertion is true when f is an increasing function.

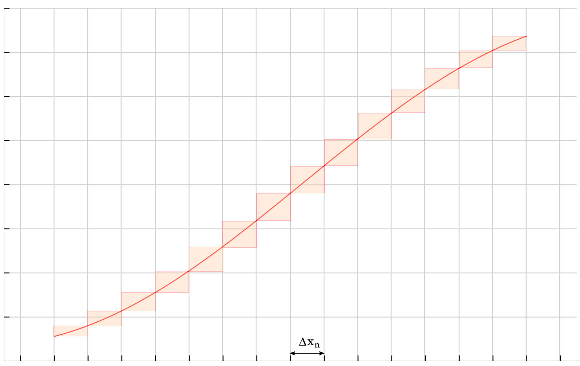


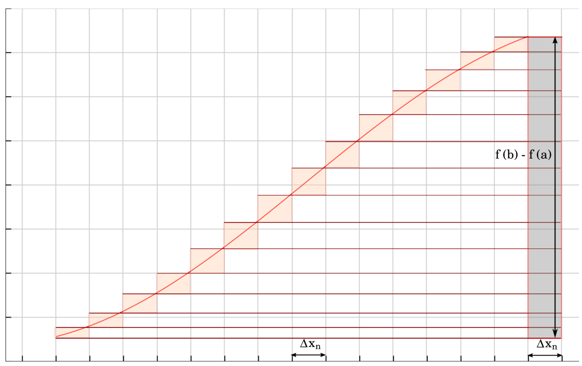


Since f is an increasing function on each subinterval I_k, the the function f has minimum value at the left endpoint and maximum value at the right endpoint. This means:

- The under estimate Riemann sum is the left endpoint Riemann sum $\sum_{k=1}^{n} f(x_{k-1})\,\Delta x_n$.
- The over estimate Riemann sum is the right endpoint Riemann sum $\sum_{k=1}^{n} f(x_k)\,\Delta x_n$.

The difference of the over estimate and under estimate Riemann sums (for n equal length subintervals) are the shaded rectangles in the left figure. These rectangles all have base length Δx_n. They can be slided to produce a vertical column of base Δx_n and height $f(b) - f(a)$.





The difference is:

$$\big(\,f(b) - f(a)\,\big)\,\Delta x_n = \big(\,f(b) - f(a)\,\big)\,\frac{b-a}{n}$$

As $n \to \infty$, the factor $\frac{b-a}{n}$ in the right hand side has limit zero, so the difference of the over and under estimate has limit zero. This means the limits of the under and over estimates must exist and have the same value.

The logic in changing notation from Riemann sum to definite integral is:

$$\sum_{k=1}^{n} f(x_k)\,\Delta x_n \quad \text{is converted to} \quad \int_a^b f(x)\,dx$$

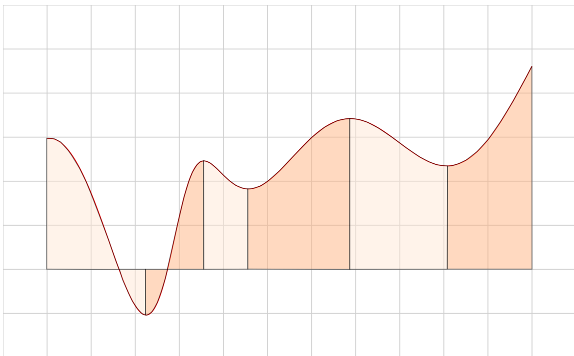
Basic properties of definite integrals.

- If the function f is has domain $[a, c]$, and the function is integrable on the subintervals $[a, b]$ and $[b, c]$, then the function is integrable on the entire interval $[a, c]$, and

$$\int_a^b f(x)\,dx \;+\; \int_b^c f(x)\,dx \;=\; \int_a^c f(x)\,dx$$

  This property is quite useful. A function which is increasing (or decreasing) on an interval is integrable. If the domain $[r, s]$ of a function f can be partitioned into (a finite number of) subintervals on which the function is either increasing or decreasing, then the function is integrable on each subinterval, and therefore integrable on the entire interval $[r, s]$.

  In the figure, the interval has been partition into 6 subintervals on which the function is decreasing or increasing. On each subinterval, the function is integrable, so the function is integrable on the entire interval.



  Most common functions such as polynomials, exponential, trignometric, rational, power and their combinations and composites satisfy the above property, and are therefore integrable on any intevral $[a, b]$ on which they are defined (meaning no undefined points such as an infinite limit).

- If f and g are two functions integrable on the domain $[a, b]$, then:
  - Their sum $f + g$ is integrable, and

$$\int_a^b (\,f(x) + g(x)\,)\,dx \;=\; \int_a^b f(x)\,dx \;+\; \int_a^b g(x)\,dx\ .$$

  - If we multiply by a constant C, we have

$$\int_a^b C\,f(x)\,dx \;=\; C\int_a^b f(x)\,dx\ .$$

- If f and g are two functions integrable on the domain $[a, b]$, and $f(x) \le g(x)$ throughout the interval, then

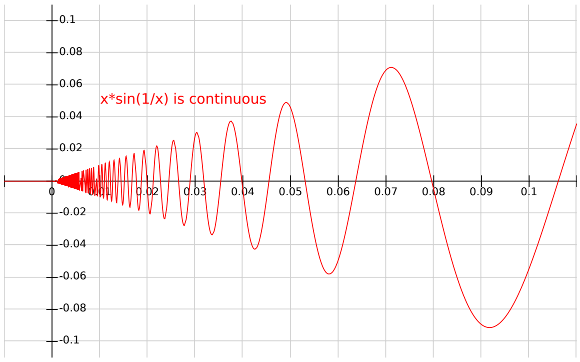
$$\int_a^b f(x)\,dx \;\le\; \int_a^b g(x)\,dx\ .$$

- $f(x)\,dx = -\int_b^a f(x)\,dx$, and $\int_a^a f(x)\,dx = 0$.
- If the function values are ≤ 0, then

$$\int_a^b f(x)\,dx \;=\; \text{minus area between graph of } f \text{ and } x\text{-axis (on interval } [a, b]).$$

- If a function f, with domain $[a, b]$, is integrable, then it is also integrable on any inside subinterval $[c, d]$.
- If a function f, with domain $[a, b]$, is continuous, then it is integrable. For common continuous functions, with domain $[a, b]$, we can decompose the interval into a finite sumber of subintervals on which the function is increasing or decreasing and the 1st property applies to show f is integrable.

  But there are continuous function on an interval $[a, b]$ for which it is not possible to divide the interval in a fincite number of subintervals of increase and decrease. An example is the function $f(0) = 0$, and $f(x) = x\ \sin(1/x)$ for $0 < x < 1$. It is continuous on $[0, 1]$, but $[0, 1]$ can be divided into a finite number of subintervals where the function is only increasing or decreasing.



Example of a function which is not integrable.
Take the following function with domain $[0, 1]$:

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is an irrational number} \\ 1 & \text{when } x \text{ is an rational number} \end{cases}$$

On any subinterval, the minimum value of the function is 0 and the maximum value is 1. This means any over estimate Riemann summ equal 1 and any under estimate Riemann sum equals 0. For equal length subintervals, the over estimate Riemann sums has limit 1 and the under estimate Riemann sums has limit 0. So, the function is not integrable.