Techniques of integration - substitution.

The powerful statement of the fundamental theorems of calculus is that we can compute the definite integral of a function f over an interval $[a, b]$ by finding an anti-derivative F of f. Using the rules for derivatives in 'reverse' we can obtain ways (techniques) to find anti-derivatives. The two most used derivative rules are the chain rule and the product rule. In 'reverse' these two rules give two techniques to find anti-derivatives.

- reversing chain rule gives **substitution method** for finding anti-derivatives.
- reversing product rule gives **integration by parts method**.

Chain rule: If g and h are two differentiable functions and we compose them to get $F(x) = h(g(x))$, then:

$$F(x) \;=\; h(\,g(x)\,) \quad\Longrightarrow\quad F'(x) \;=\; h'\,(g(x))\,g'(x)$$

Therefore, if we can write a function $f(x)$ in the form

$$f(x) \;=\; h'(g(x))\,g'(x) \quad \text{then} \quad F(x) = h(g(x)) \;\; \text{is an anti-derivative}$$

In terms of differentials, if $u = g(x)$, then $du = g'(x)\,dx$, and so for any $h(u)$, we have

$$\int h(u)\,g'(x)\;dx \;=\; \int h(u)\;du$$

Examples.

- Find anti-derivatives of $\dfrac{x}{1+x^2}$.

  Solution 1. We know $h(u) = \ln(u)$ has derivative $\frac{1}{u}$. To match with $\dfrac{x}{1+x^2}$, we take $u = 1 + x^2$ (so $g(x) = 1 + x^2$). Then, $h(g(x))$ has derivative

  $$\left(\;\ln(\,1+x^2\,)\;\right)' \;=\; \frac{1}{1+x^2}\,2x \;=\; \frac{2x}{1+x^2}\;;$$

  so, $\dfrac{1}{2}\ln(\,1+x^2\,)$ is a anti-derivative of $\dfrac{x}{1+x^2}$, and thereforee, the general anti-derivative is $\dfrac{1}{2}\ln(\,1+x^2\,) \;+\; C$.

Solution 2. We wish to find $\int \frac{x}{1+x^2}\,dx$. We take $u = (1+x^2)$, so $du = 2x\,dx$, then

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,x\,dx = \int \frac{1}{u}\,\frac{1}{2}du$$

$$= \frac{1}{2}\int \frac{1}{u} = \frac{1}{2}\ln(u) + C$$

$$= \frac{1}{2}\ln(1+x^2) + C$$

- Find $\int e^x\sqrt{3+e^x}\,dx$. We take $u = (3+e^x)$, so $du = e^x\,dx$. Then

$$\int e^x\sqrt{3+e^x}\,dx = \int \sqrt{3+e^x}\,e^x\,dx = \int \sqrt{u}\,du$$
$$= \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(3+e^x)^{\frac{3}{2}} + C\,.$$

Limits of integration and substitution.

In a definite integral $\int_a^b f(x)\,dx$, the endpoints a and b of the interval are sometimes called the limits of the integration. Under subsitution, the limits of the inetgration change.

Examples.

- Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{(\cos(x))^2}\,dx$. Take $u = \cos(x)$, so that $du = -\sin(x)\,dx$. Then,

$$x = 0 \implies u = \cos(0) = 1 \quad \text{and} \quad x = \frac{\pi}{4} \implies u = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}\;;$$

so,

$$\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{(\cos(x))^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin(x)\,dx}{(\cos(x))^2} = \int_1^{\frac{1}{\sqrt{2}}} \frac{-du}{u^2}$$

$$= \int_{\frac{1}{\sqrt{2}}}^1 \frac{du}{u^2} = \left(-\frac{1}{u}\right)\Big|_{\frac{1}{\sqrt{2}}}^1$$

$$= -\frac{1}{1} - (-\frac{1}{\frac{1}{\sqrt{2}}}) = -1 + \sqrt{2}$$

- Evaluate $\displaystyle\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25\,x^2 - 1}}$. We use two subsititutions to determine the definite.

1st substituion. Take $u = 25\,x^2 - 1$, so $du = 50x\,dx$. Then,

$$u + 1 = 25\,x^2 \quad\Longrightarrow\quad \frac{dx}{x} = \frac{50\,x\,dx}{50\,x^2} = \frac{du}{2\,(u+1)}$$
$$\Longrightarrow\quad \frac{dx}{x\sqrt{25x^2-1}} = \frac{du}{2\,(u+1)\,u^{\frac{1}{2}}}\,,$$

and

$$x = \frac{2}{5\sqrt{3}} \quad\Longrightarrow u = 25\,(\frac{2}{5\sqrt{3}})^2 - 1 = \frac{4}{3} - 1 = \frac{1}{3}\,, \quad \text{and}$$
$$x = \frac{2}{5} \quad\Longrightarrow u = 25\,(\frac{2}{5})^2 - 1 = 4 - 1 = 3\,;$$

so,

$$\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25\,x^2 - 1}}\,dx = \frac{1}{2}\int_{\frac{1}{3}}^{3} \frac{du}{(u+1)\,u^{\frac{1}{2}}}\,.$$

2nd substitution. Take $u = v^2$, so $du = 2v\,dv$. Then,

$$\frac{du}{u^{\frac{1}{2}}} = \frac{2v\,dv}{v} = 2\,dv \quad\Longrightarrow\quad \frac{du}{(u+1)\,u^{\frac{1}{2}}} = \frac{2\,dv}{v^2+1}\,;$$

so,

$$\frac{1}{2}\int_{\frac{1}{3}}^{3} \frac{du}{(u+1)\,u^{\frac{1}{2}}} = \frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{2\,dv}{v^2+1} = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dv}{v^2+1} = \arctan(v)\,\Big|_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$
$$= \arctan(\sqrt{3}) - \arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{3} = \frac{\pi}{6} = \frac{\pi}{6}$$

so,

$$\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25\,x^2 - 1}} = \frac{1}{2}\int_{\frac{1}{3}}^{3} \frac{du}{(u+1)\,u^{\frac{1}{2}}} = \frac{\pi}{6}\,.$$

Applications of integrals.

If f is a integrable function with domain $[a, b]$, the number

$$\frac{1}{b-a} \int_a^b f(x)\, dx$$

is the average value of the function on the interval.

Examples.

- A hiking trail has elevation in feet given by

$$f(x) \;=\; 60\,x^3 \;-\; 650\,x^2 \;+\; 1200\,x \;+\; 4500 \quad \text{for } 0 \le x \le 5 \text{ (miles).}$$

Find the average elevation of the trail. The integral is

$$\begin{aligned} \int_0^5 f(x)\, dx &= 15\,x^4 \;-\; \frac{650}{3}\,x^3 \;+\; 600\,x^2 \;+\; 4500\,x \,\Big|_0^5 \\ &= 15 \cdot 5^4 \;-\; \frac{650}{3} \cdot 5^3 \;+\; 600 \cdot 5^2 \;+\; 4500 \cdot 5 \\ &= 19791\,\frac{2}{3} \quad \text{(feet)} \end{aligned}$$

Therefore, the average elevation is $\dfrac{19791\,\frac{2}{3}}{5} \;=\; 3958\,\frac{1}{3}$ (feet).

- Find the average value of the function $\sin(x)$ on the interval $[0, \pi]$.

In a previous example, we computed $\int_0^\pi \sin(x)\, dx \;=\; 2$; therefore, the average value of sin on $[0, \pi]$ is:

$$\frac{1}{\pi} \int_0^\pi \sin(x)\, dx \;=\; \frac{2}{\pi}\,.$$

- Integral Mean Value Theorem. The term mean is often used in place of the term average. We apply the derivative mean value to the area function of a continuous function f:

Suppose f is continuous on the interval $[a, b]$ and $A(s) \;=\; \int_a^s f(x)\, dx$; so,

$$A(a) \;=\; 0\;, \quad A(b) \;=\; \int_a^b f(x)\, dx\;, \quad A'(s) \;=\; f(x)\,.$$

The derivative Mean Value Theorem says the slope of the secant line from $P \;=\; (a, A(a))$ to $Q \;=\; (b, A(b))$ is equal to the value of $A' = f$ at some interior point c, So

$$\frac{A(b) \;-\; A(a)}{b \;-\; a} \;=\; A'(c) \quad \text{some interior point } c$$

$$\frac{1}{b \;-\; a} \int_a^b f(x)\, dx \;=\; f(c)\,.$$

So the average value of a continuous function f on an interval $[a, b]$ will equals its value at some point in the interior of the interval.

Application of integrals to finding volume - solids of revolutions.

Examples.

- Consider an inverted cone with height h and top radius r. The function $x = (r/h)\,y$ gives the radius of a circular cross section at height y. The volume of an infinitesimal cylinder of height dy and base radius x is

$$dV = \pi\, x^2\, dy = \pi\,(\frac{r}{h}\,y\,)^2\, dy\ ;$$

so,

$$\begin{aligned}\text{Volume of cone} &= \int_0^h \pi\,(\frac{r}{h}\,y\,)^2\,dy = \pi\,(\frac{r}{h})^2 \int_0^h y^2\,dy \\ &= \pi\,(\frac{r}{h})^2\,(\frac{y^3}{3}\,)\Big|_0^h = \pi\,(\frac{r}{h})^2\,(\frac{h^3}{3} - 0\,) = \pi\,r^2\,h\,\frac{1}{3} \\ &= \frac{1}{3}\cdot \text{area of base}\cdot\text{height}\ .\end{aligned}$$

- The volume of a sphere of radius r can be calculated as follows: Tkae the graph of the function $f(x) = \sqrt{r^2 - x^2}$ with domain $[-r, r]$ and revolve the graph around the x-axis to poduce a sphere. The volume of a infinitesimal cyclinder of 'horizontal height' dx and radius $f(x)$ is

$$dV = \pi\, f(x)^2\, dx = \pi\,(\,r^2 - x^2\,)\,dx;$$

so,

$$\begin{aligned}\text{Volume of cone} &= \int_{-r}^{r} \pi\,(\,r^2 - x^2\,)\,dx = \pi\,(\,r^2\,x - \frac{x^3}{3}\,)\Big|_{-r}^{r} \\ &= \pi\Big(\,(\,r^2\,r - \frac{r^3}{3}\,) - (\,r^2\,(-r) - \frac{(-r)^3}{3}\,)\Big) \\ &= \frac{4}{3}\,\pi\,r^3\ .\end{aligned}$$