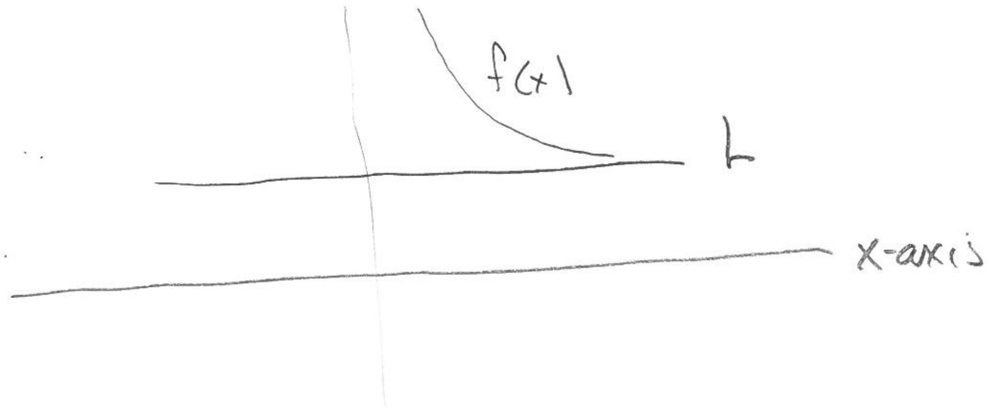
Horizontal asymptote

If $\lim\limits_{x \to \infty} f(x) = L$ we say horizontal line $y = L$ is a horizontal asymptote.



Another way to think about this..

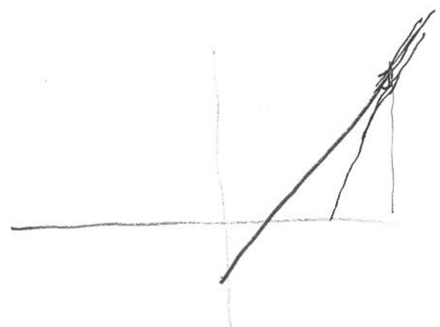
Think $y = L$ as (constant) function.

$$\lim_{x \to \infty} \underbrace{(f(x) - L)}_{\text{difference in the two functions}} = 0$$

difference in the two functions $f(x)$, $y = L$ has limit 0 as $x \to \infty$

WW3 #16 $\lim\limits_{x \to \infty} \left(\sqrt{x^2 - 9x + 1} - x\right) = -\frac{9}{2}$

same as $\lim\limits_{x \to \infty} \left(\underbrace{\left(\sqrt{x^2 - 9x + 1}\right)}_{\text{1st function}} - \underbrace{\left(x - \frac{9}{2}\right)}_{\text{2nd function}}\right) = 0$



Say line $y = \left(x - \frac{9}{2}\right)$ is a slant asymptote of function $\sqrt{x^2 - 9x + 1}$

Growth of functions. If f, g are two functions, we say they have same relative growth if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \quad (\neq 0).$$

If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$, we say g grows faster than f.

Examples ① $f(x) = \sqrt{x^2 - 9x + 1}$, $g(x) = x$, $\lim_{x \to \infty} \frac{\sqrt{x^2 - 9x + 1}}{x} = 1$

Both f, g are growing towards ∞ as $x \to \infty$, but their relative growth is the same

② $f(x) = x$, $g(x) = x^2$, $\frac{f(x)}{g(x)} = \frac{x}{x^2} = \frac{1}{x}$ $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

Parabola $y = x^2$ grows faster than line $y = x$.

③ If $m < n$, then $y = x^n$ grows faster than $y = x^m$

(5) $f(x) = x^5 + \sin(x) + 9999$ $\qquad \dfrac{f(x)}{g(x)} = \dfrac{1}{x} + \dfrac{\sin(x)}{x^6} + \dfrac{9999}{x^6}$

$g(x) = x^6$

as $x \to +\infty$ limit is $0+0+0=0$

So $g(x) = x^6$ grows faster.

<u>Continuity</u> Whether $\lim\limits_{x \to a} f(x)$ exists or does not exists

has <u>NOTHING</u> to do with value of f at input a. In fact

a may not be domain of f.

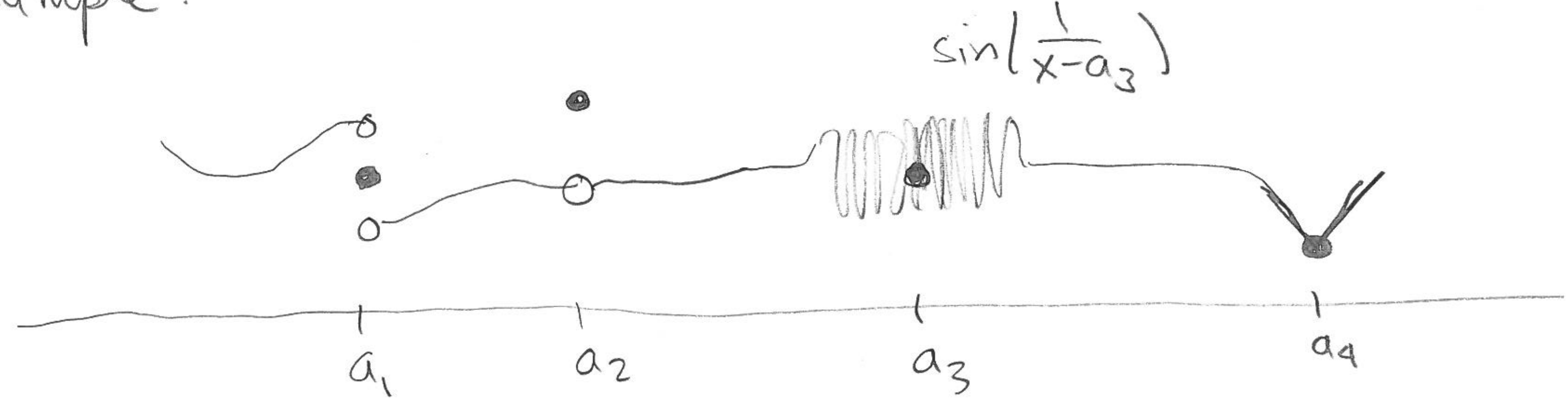
For many common functions on an interval <u>I</u> we have

(1) $\lim\limits_{x \to a} f(x)$ exists.

(2) a is an allowed input and value of $\lim\limits_{x \to a} f(x)$ equals $f(a)$.

When this happens, say <u>f is continuous at input a</u>.

Example.



a_1: $\lim\limits_{x\to a_1} f(x)$ does not exist, so cannot be $f(a)$.

a_2: $\lim\limits_{x\to a_2} f(x) = L$ exists, but $L \neq f(a)$.

a_3: $\lim\limits_{x\to a_3} f(x) = \text{DNE}$ | a_4: $\lim\limits_{x\to a_4} f(x) = M = f(a_4)$

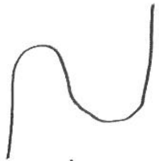
At all other inputs a, the function is continuous

A function f is continuous everywhere if continuous at ALL inputs

Intuition A function is continuous if we can draw graph without lifting pencil off paper.

Examples. Common function such as polynomial, sin, cos are continuous functions everywhere.

quadr



cubic

sine.

WW3 # 18

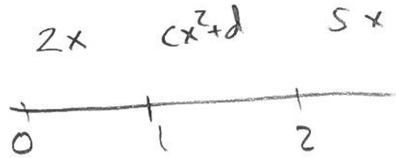
$$f(x) = \begin{cases} 2x & x < 1 \\ cx^2 + d & 1 \le x < 2 \\ 5x & 2 \le x \end{cases}$$

continuous on $x < 1$

continuous $1 \le x < 2$

continuous on $2 \le x$

Pick c and d so f is continuous <u>EVERYWHERE</u>.



Need c, d so $\lim\limits_{x \to 1^-} 2x = \lim\limits_{x \to 1^+} cx^2 + d$.

$2 \cdot 1 = c \cdot 1^2 + d \qquad 2 = c + d.$

Also $\lim\limits_{x \to 2^-} (cx^2 + d) = \lim\limits_{x \to 2^+} 5x$

$c \cdot 2^2 + d = 5 \cdot 2. \qquad 10 = 4c + d.$

Two equations Two unknowns.

$10 - 2 = (4-1)c + 0d$

$8 = 3c \quad c = \frac{8}{3}, d = -\frac{2}{3}$

Derivative (fancy name for tangent slope).

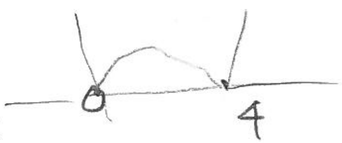
Assume f is function with domain interval I

Take $a \in I$. If the limit of secant slopes base at a.

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = L \text{ exists} \quad \left(\begin{array}{l}\text{physically this} \\ \text{tangent slope at } a\end{array}\right)$$

We say f differentiable at input a.

Ex. $f(x) = \left|\frac{1}{4}x^2 - x\right| = \begin{cases} \frac{1}{4}x^2 - x & \text{when } x<0 \text{ OR } 4 \le x \\ -\left(\frac{1}{4}x^2 - x\right) & \text{when } 0<x<4. \end{cases}$



From before

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \begin{cases} \frac{1}{2}a - 1 & a<0 \text{ OR } 4<a \\ -\frac{1}{2}a + 1 & 0<a<4 \\ \text{DNE} & a=0,\ a=4 \end{cases}$$

This function differentiable at all inputs EXCEPT $a=0$, $a=4$.

Another physical interpretation of derivative.

Think of point moving along a line, with position function $p(t)$ $t = $ time.

$p(t) - p(a) =$ net change in position during interval $[a, t]$. (or $[t, a]$).

$t - a =$ net change in time

The ratio $\dfrac{p(t) - p(a)}{t - a} =$ average speed during interval $[a, t]$.

$\displaystyle\lim_{t \to a} \frac{p(t) - p(a)}{t - a} =$ instantaneous speed, at time a.

Example. $p(t) = t^2$, $\dfrac{p(t) - p(a)}{t - a} = \dfrac{t^2 - a^2}{t - a} = \dfrac{(t+a)\cancel{(t-a)}}{\cancel{(t-a)}} = t + a.$

So $\displaystyle\lim_{t \to a} \frac{p(t) - p(a)}{t - a} = a + a = 2a.$