# Derivative as instantaneous rate of change

Let a point move along an axis with its position given by the function $p(t)$


If $a<t$, then

$$\frac{p(t)-p(a)}{t-a} = \frac{\text{change in position}}{\text{time interval}} = \text{average speed}$$

Take limit $\lim_{t \to a} \frac{p(t)-p(a)}{t-a} = \text{instantaneous speed.}$

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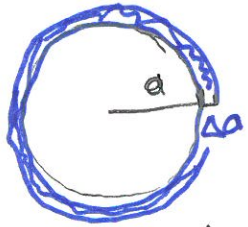
More generally if $Q$ is a quantity that is a function of a variable $s$, we derivative

$$\lim_{s \to a} \frac{Q(s)-Q(a)}{s-a} = \text{instantaneous rate of change.}$$

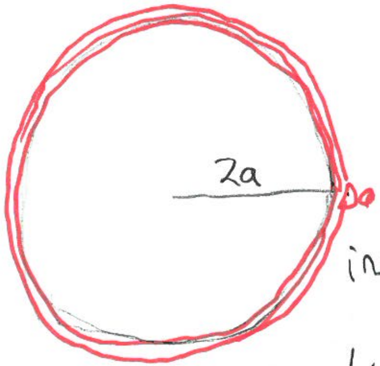
Examples ① Area of circular disk $A(r) = \pi r^2$

$$\lim_{r \to a} \frac{A(r) - A(a)}{r-a} = \lim_{r \to a} \frac{\pi r^2 - \pi a^2}{r-a} = \lim_{r \to a} \frac{\pi \cancel{(r-a)}(r+a)}{\cancel{(r-a)}} = \pi(a+a) = 2\pi a$$




instantaneous rate of change of A when radius is a is $2\pi a$.

Increase in area when we go from $r=a$, to $r=a+\Delta a$ is $(2\pi a)\cdot \Delta a$.




instantaneous rate of change is $(2\pi)(2a) = 4\pi a$.

When we increase radius from $r = 2a$ to $r = 2a + \Delta a$, the increase in area is approximate $(4\pi a)\cdot(\Delta a)$

② Formula to change temperature from $F^\circ$ to $C^\circ$.

$$C(F) = \frac{5}{9}(F-32)$$

$$\lim_{F \to a} \frac{C(F) - C(a)}{F-a} = \lim_{F \to a} \frac{\frac{5}{9}(F-32) - \frac{5}{9}(a-32)}{F-a} = \frac{5}{9}$$ constant rate of change.

No matter what is the temp. If F changes by 1 degree, C changes by $\frac{5}{9}$ degree.

Derivative at specific input VS Derivative function

lim / number

function

input $a$.

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

We take all inputs $a$ for which the derivative exists, and we define / set the function $f'$

$$a \xrightarrow{f'} \text{derivative value at } a$$

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Example $f(x) = x^{1/3}$ cube root function domain is $\mathbb{R}$.

Derivative at input $a$

$$(r-s)(r^2+rs+s^2) = (r^3-s^3)$$

$$r = x^{1/3}, \quad s = a^{1/3} \quad (x-a)$$

$$\frac{f(x)-f(a)}{x-a} = \frac{(x^{1/3}-a^{1/3})}{x-a} \cdot \frac{(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}{(x^{2/3}+x^{1/3}a^{1/3}a^{2/3})}$$

$$= \frac{(\cancel{x-a})}{(\cancel{x-a})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$

$$\lim_{x \to a} (\ ) = \lim_{x \to a} \frac{1}{(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})} = \frac{1}{(a^{2/3}+a^{1/3}a^{1/3}+a^{2/3})} = \frac{1}{3} a^{-2/3}$$

$(a \neq 0)$

WW4 #4. Graph of function.
Select correct graph for its derivative function

#5 Math graphs of 3 functions with $f, f', f''$

$(\text{red})' = \text{green}$. $(\text{blue})' = \text{red}$.

$\underset{\text{blue}}{f},\ \underset{\text{red}}{f'},\ \underset{\text{green}}{f''}$.

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Basic derivatives and basic derivative rules.

$f(x) = x^n$ n positive integer.

$$\frac{f(x)-f(a)}{x-a} = \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1}$$

so

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \underbrace{a^{n-1} + a^{n-2} \cdot a + \dots + a a^{n-2} + a^{n-1}}_{n \text{ terms all equal } a^{n-1}}$$

$$= n a^{n-1} \quad \Big| \quad f'(a) = n a^{n-1}.$$

$$\begin{array}{r l}
 & x^{n-1} + x^{n-2}a + \cdots \qquad + x0^{n-2} + a^{n-1} \\
x-a \;) & x^n + 0x^{n-1} + \cdots \qquad + 0x - a^n \\
-( & x^n - ax^{n-1} \\
 & \overline{\quad ax^{n-1}} \\
 & ax^{n-1} - a^2 x^{n-2} \\
 & \overline{\quad a^2 x^{n-2}}
\end{array}$$

Remainder $= 0$

In fact for any number $r$, if $f(x) = x^r$, then

$$f'(a) = r a^{r-1}.$$

Example $f(x) = \frac{1}{x} = x^{-1}$, has derivative function

$$f'(a) = (-1) a^{-1-1} = -1 a^{-2} = -\frac{1}{a^2}$$

<u>Basic derivative rule.</u> If $f, g$ are two functions with derivatives, then

$$(f \pm g)'(a) = f'(a) \pm g'(a) \quad \text{sum/minus}$$

$$(C \cdot f)'(a) = C \cdot f'(a) \quad C \text{ constant}$$

These follow/proved ~~from~~ from corresponding rules for limits.

Example. Find derivative function of

$f(x) = x^3 - 2x^2 + 1$, by roles and basic fact

$$f'(a) = 3a^2 - 2\cdot(2a^1) + 0$$
$$= 3a^2 - 4a.$$

To find "valley", find we $f'(a) = 0$

$$0 = 3a^2 - 4a = a(3a-4)$$
$$a = 0,\quad a = \frac{4}{3}$$

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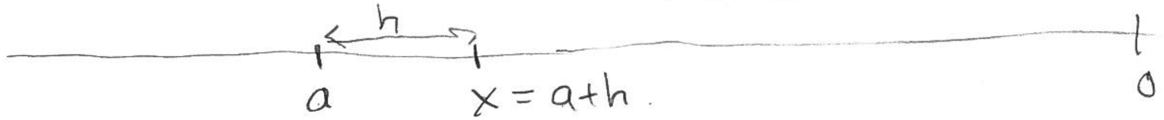
Other ways to say limit for derivative.

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \quad \text{SAME AS} \quad \lim_{h\to 0} \frac{f(a+h)-f(a)}{(a+h)-a}$$




$f(x) = f(a+h)$ $\quad$ $f(a) = f(a)$

$x - a = h.$