Week 5 F L03 2pm

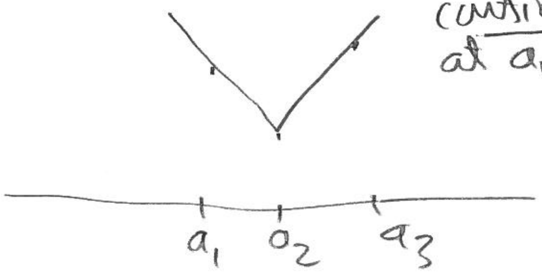
Continuous VS Differentiable

Both. The approach point a is in the domain.

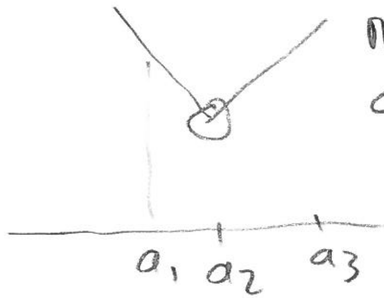
$\lim\limits_{x \to a} f(x) = f(a)$

$\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$ exists.

Pictorial example



continuous at a_1, a_2, a_3



not differentiable at a_2

Continuous at a $\not\Rightarrow$ differentiable at a

does not mean

Other way is true continuous at a $\Leftarrow$ differentiable at a

implies.

Differentiable at a means $\lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a} = L$ exists.

But we clearly have $\lim\limits_{x \to a} (x-a) = 0$.

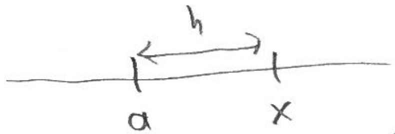
The product $\left(\dfrac{f(x)-f(a)}{x-a}\right) \cdot (x-a)$ must therefore have limit $L \cdot 0 = 0$.

This says $\lim\limits_{x \to a} (f(x)-f(a)) = 0$ so $\lim\limits_{x \to a} f(x) = f(a)$

Different way to say derivative limit.

$\lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a} = L$ exists SAME AS $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h} = L$ exists.

2nd way to say derivative exists.



$x = a+h$

$h = x-a$.

Derivative of sine, cosine.

$$\sin(A+B) = \cos(A)\sin(B) + \sin(A)\cos(B)$$
$$\quad\ \ a\quad h$$

$$\frac{\sin(a+h) - \sin(a)}{h} = \frac{\cos(a)\sin(h) + \sin(a)\cos(h) - \sin(a)}{h}$$

$$= \cos(a)\left(\frac{\sin(h)}{h}\right) + \sin(a)\left(\frac{\cos(h)-1}{h}\right).$$

As $h \to 0$, we see from squeeze theorem $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$.

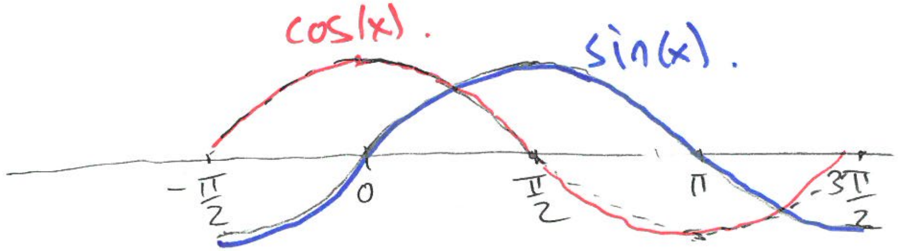
also from squeeze theorem $\lim_{h\to 0} \left(\frac{\cos(h)-1}{h}\right) = 0$.

0

$$\text{Therefore } \lim_{h\to 0} \frac{\sin(a+h)-\sin(a)}{h} = \cos(a)\cdot 1 + \cancel{\sin(a)}\cdot 0$$
$$= \cos(a)$$

Tangent slope of $f(x) = \sin(x)$ at input a is $\cos(a)$

$$f'(a) = \cos(a).$$



$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$A = a,\ B = h$

Derivative of cosine

$$\frac{\cos(a+h) - \cos(a)}{h} = \text{secant slope} = \ldots$$

$$\lim_{h \to 0} \left(\frac{\cos(a+h) - \cos(a)}{h} \right) = -\sin(a)$$

Derivative of $f(x) = \cos(x)$ is $f'(a) = -\sin(a)$.

Two rules for derivatives (PRODUCT / QUOTIENT)

Suppose f, g are differentiable.

PRODUCT $(f \cdot g)' = f'g + fg'$ $\quad\quad \left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$

QUOTIENT $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ $\quad\quad \left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$

assume $g(a) \neq 0$.

Why PRODUCT rule?

$$\frac{f(a+h)g(a+h) - f(a)g(a)}{h} = \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h}$$

$$= \left(\frac{f(a+h) - f(a)}{h}\right) g(a+h) \quad + \quad f(a)\left(\frac{g(a+h) - g(a)}{h}\right)$$

As $h \to 0$, we have $\left(\frac{f(a+h)-f(a)}{h}\right) \to f'(a)$, $g(a+h) \to g(a)$

$f(a)$ constant, $\frac{g(a+h)-g(a)}{h} \to g'(a)$.

we get limit is $f'(a)g(a) + f(a)g'(a)$ The product rule

Reasoning of QUOTIENT RULE is similar.

WW 4 # 8 $\quad f(x) = \dfrac{12\sin(x) - 4}{\cos(x)}$

$$f'(x) = \frac{(12\sin(x)-4)'\cos(x) - (12\sin(x)-4)\cdot(\cos(x))'}{(\cos(x))^2}$$

$$= \frac{(12\cos(x))\cos(x) - (12\sin(x)-4)(-\sin(x))}{(\quad)^2}$$

$$= \frac{12(\cos x)^2 + 12(\sin x)^2 - 4\sin(x)}{(\quad)^2}$$

$$= \frac{12 - 4\sin(x)}{(\cos(x))^2}$$

#6

Given $f(-5) = 5$, $f'(-5) = -5$
$g(-5) = -4$, $g'(-5) = 4$

Find $(fg)'(-5)$.

By product rule

$$(fg)'(-5) = f'(-5)g(-5) + f(-5)g'(-5)$$
$$= (-5)(-4) + 5\cdot(4) = 20+20 = 40$$

COMPOSITION (CHAIN) RULE Suppose f, g are differentiable functions, and composition $(f \circ g)(x) = f(g(x))$ makes sense. Then derivative at input a is

$$(f \circ g)'(a) = f'(g(a))\, g'(a)$$

$b = g(a)$

$f(g(a)) = f(b)$

Why true?

$$\frac{f(g(a+h)) - f(g(a))}{h} = \frac{f(y) - f(b)}{y-b} \; \frac{y-b}{h}.$$

new variables.

$$\begin{cases} y = g(a+h) \\ b = g(a) \end{cases}$$

As $h \to 0$, $\frac{y-b}{h} = \frac{g(a+h) - g(a)}{h} \longrightarrow g'(a)$

also $h \to 0$, $y \to g(a) = b$. So $\lim_{y \to b} \left(\frac{f(y) - f(b)}{y - b} \right) = f'(b)$

WW4 #9 Find derivative of $w(r) = \sqrt{r^7+5}$

w is composition of inside function $g(r) = r^7+5$ | $g'(r) = 7r^6$

outside function $f(u) = \sqrt{u}$

$$w'(r) = \frac{1}{2}(g(r))^{-1/2}\, g'(r) = \frac{1}{2}(r^7+5)^{-1/2} \cdot 7r^6$$

#12 Find derivative of $f(x) = \sqrt{2+(\sin(x))^2}$

inside function is $g(x) = 2+(\sin(x))^2 = 2+(\sin(x))(\sin(x))$

$g'(x) = 0 + (\sin x)' \sin(x) + (\sin(x))(\sin(x))'$

$= 2\cos x \sin x$

outside function is $h(u) = \sqrt{u}$

By composition/chain rule ($f = h \circ g$)

$$f'(x) = \underbrace{\frac{1}{2}(2+(\sin x)^2)^{-1/2}}_{\text{outside derivative}} \cdot \underbrace{2\cos x \sin x}_{\text{inside derivative}}.$$

#13 Given $\frac{d}{dx}(f(3x^2)) = 8x^4$, Find $f'(x)$

$g(x) = 3x^2$ $\quad$ $(f \circ g)'(x) = 8x^4$

$$f'(3x^2) \cdot g'(x) = 8x^4$$

$$f'(3x^2) \cdot (3 \cdot 2x) = 8x^4$$

$$f'(3x^2) = \frac{4}{3}x^3$$

Set $u = 3x^2$, compute inverse $\frac{u}{3} = x^2$, so $x = \left(\frac{u}{3}\right)^{1/2}$

Substitute $\quad f'(u) = \left(\frac{4}{3}\right)\left(\frac{u}{3}\right)^{3/2}$

Now change u to x to get $f'(x)$