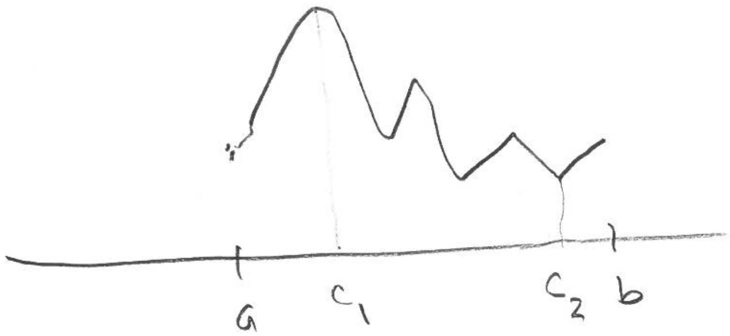
Extreme Value Theorem (EVT) If f is continuous function on a closed interval $[a,b]$, then there is an input c_1 so $f(c_1)$ is abs max, and an input c_2 so that $f(c_2)$ is abs min.



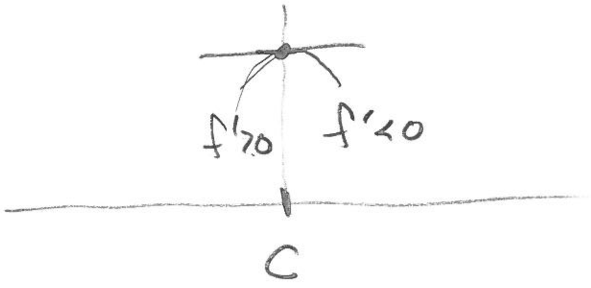
To find inputs for abs max/min, we look at set of local max/min

Local Extreme Value Theorem (LEVT) If f is continuous function on closed $[a,b]$, the local max/min pts occur in

1. $f'(c)=0$ (exists) } critical pts
2. $f'(c)$ DNE
3. endpoints.

Use LEVT to find local max/min. Usually finite set.
We then pick abs max/min from this finite set.

At critical point where $f'(c)=0$, we distinguish between local max or local min as follows.

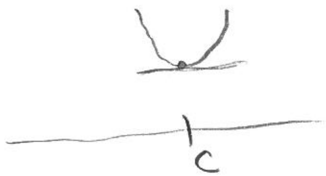


If $f'(c)=0$, but $f'(c^-)>0$, $f'(c^+)<0$
rising falling

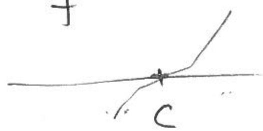
then c is local max.

Here f' switched from >0 to <0 at c.
$+$ $-$

If f' switches from $-$ to $+$ at c then c yields local min



f' $f'(c)=0$



2nd derivative test

(1) Suppose $f''(c)>0$ so f' increasing

so f' must switch from $-$ to $+$ at c so local min

(2) Suppose $f''(c)<0$, then f' decreasing

so f' switches $+$ to $-$ at c $\Rightarrow$ local max.

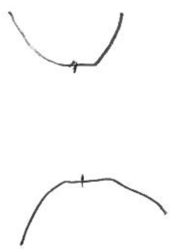
Best linear approx to f near input c is tangent line

$$T(x) = y = f(c) + f'(c)(x-c) \;=\; f(c) \text{ since } f'(c)=0$$ | c critical pt

Best quadratic approximation

$$y = f(c) + \underbrace{\cancel{f'(c)}}_{0}(x-c) + \frac{f''(c)}{2}(x-c)^2$$

$$= f(c) + \frac{f''(c)}{2}(x-c)^2 \quad \text{shape is}$$



$f''(c)>0$

$f''(c)<0$

So $f''(c)>0 \Rightarrow$ upward parabola $\Rightarrow$ min

$f''(c)<0 \Rightarrow$ downwar parabola $\Rightarrow$ max

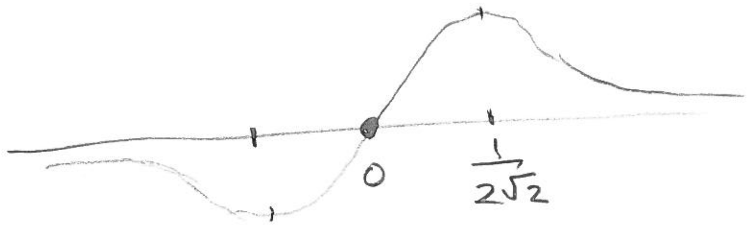
WW6 #10 $f(x) = 2x\, e^{-4x^2}$ domain is $(-\infty, \infty)$

Find critical pts. Use 2nd derivative test to determine local max/min

Note ① f is an <u>ODD</u> function ($f(-x) = -f(x)$)

As $x \to \pm\infty$ $2x$ vs e^{-4x^2}

we see $\lim_{x \to \pm\infty} f(x) = 0$



Rough graph.

② f ODD $\Rightarrow f'$ EVEN

$\Rightarrow f''$ ODD.

Find critical pt:

$$f'(x) = 2 \cdot \left\{ 1 \cdot e^{-4x^2} + x \cdot e^{-4x^2} \cdot (-4 \cdot 2x) \right\} \qquad 1 + x(-8x)$$

$$= 2\, e^{-4x^2} \left\{ 1 - 8x^2 \right\}. \qquad f'(x) = 0 \text{ when } 1 - 8x^2 = 0$$

$8x^2 = 1$, $x^2 = \frac{1}{8}$

$x = \pm \frac{1}{2\sqrt{2}}$

Use 2nd derivative test to determine/check whether critical pt is local max or min

$$f'(x) = 2e^{-4x^2}(1-8x^2)$$

$$f''(x) = 2\cdot\left\{e^{-4x^2}(-4\cdot 2x)(1-8x^2) + e^{-4x^2}(0 - 8\cdot 2x)\right\}$$

$$= 2e^{-4x^2}(-8)\left\{x(1-8x^2) + 2x\right\} \qquad 1-8x^2+2$$

$$= 2e^{-4x^2}(-8)\,x\left\{3-8x^2\right\}.$$

$$\left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{8}$$

At critical pt $\frac{1}{2\sqrt{2}}$ we get

$$3-1=2$$

$$f''\left(\frac{1}{2\sqrt{2}}\right) = 2e^{-4\left(\frac{1}{2\sqrt{2}}\right)^2}(-8)\left(\frac{1}{2\sqrt{2}}\right)\left\{3 - 8\cdot\frac{1}{8}\right\}$$

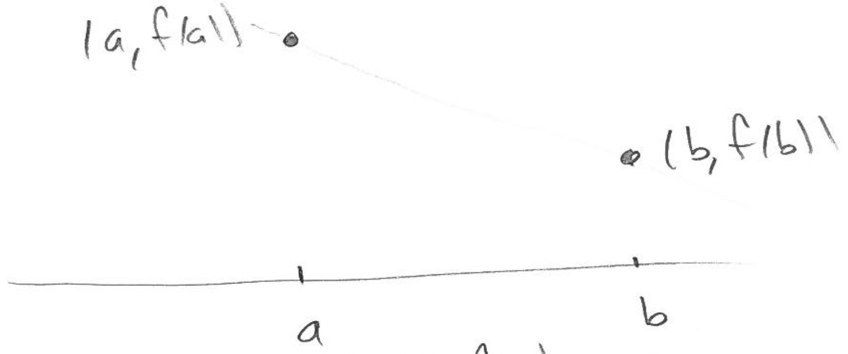
$$< 0 \quad \text{(because of } -8\text{)}.$$

By 2nd derivative test $\Rightarrow$ critical pt $\frac{1}{2\sqrt{2}}$ is a local max.

For critical pt $\frac{-1}{2\sqrt{2}}$, we get $f''\left(\frac{-1}{2\sqrt{2}}\right) = -f''\left(\frac{1}{2\sqrt{2}}\right) > 0$

$\Rightarrow \frac{-1}{2\sqrt{2}}$ is local min.

Mean Value Theorem Suppose f is differentiable function on a closed interval $[a,b]$



Secant slope $(a, f(a))$ to $(b, f(b))$ is $\frac{f(b)-f(a)}{b-a}$. Then

$\frac{f(b)-f(a)}{b-a}$ = equals the value of f' at some interior point

Very important uses of MVT

1. $f' > 0$ on interval $\Rightarrow$ f increasing
2. $f' < 0$ on —— $\Rightarrow$ f decreasing
3. $f' = 0$ (zero throughout interval) $\Rightarrow$ f is constant.
4. MVT used to prove L'hopital's rule.

$f' > 0$ on closed interval $\Rightarrow$ f increasing

"Proof by MVT"

Secant slope $(x_1, f(x_1))$ to $(x_2, f(x_2))$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \text{MVT there is point } b \text{ in the interior so that} = f'(b)$$

But $f(x_2) - f(x_1) = \underbrace{(x_2 - x_1)}_{\text{positive}} \cdot \underbrace{f'(b)}_{\text{positive since } f' > 0}$

> 0

so $f(x_2) > f(x_1)$.

Proof that if $f' \equiv 0$ on interval $[a,b]$, then f is constant.

Look at secant slope from $(x_1, f(x_1))$ to $(x_2, f(x_2))$. It is

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \overset{MVT}{=}$$ equals $f'(b)$ for some interior pt b

$$= f'(b) = 0$$ since $f' \equiv 0$.

So $f(x_2)-f(x_1)=0$ so $f(x_1)=f(x_2)$. f is constant.

WW6 # 7 f differentiable on interval $[3,7]$

and $-3 \le f'(x) \le 2$.

MVT says

$$\frac{f(7)-f(3)}{7-3} = \text{MVT equals } f'(b) \text{ at some interior point.}$$

Since $-3 \le f(b) \le 2$ we get

$$-3 \le \frac{f(7)-f(3)}{4} \le 2$$

So $$-12 \le f(7)-f(3) \le 8.$$

The MVT is a consequence of EVT and LEVT.

Look at the function which is difference

$f(x) - s(x)$ = original function − secant line

This difference is continuous differentiable function

EVT $\Rightarrow$ has abs max (which is of course local max)

LEVT $\Rightarrow$ at abs max, the derivative must be zero

$0 = (f(b) - s(b))' = f'(b)$ − secant slope.

So $f'(b)$ = secant slope.