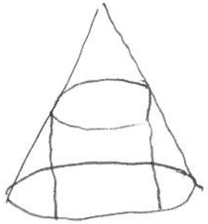
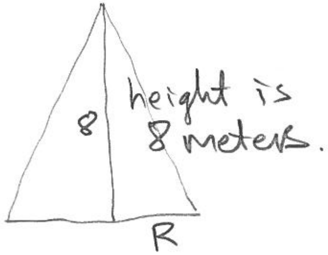
<u>Optimization</u> We have differentiable (so continuous) function, we seek its max/min.

WW7 #1





Volume is 6 cubic meters

$\frac{1}{3}$ area base $\cdot$ height $= 6 \ m^3$

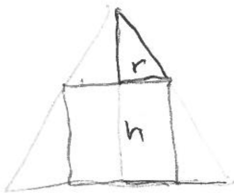
$\frac{1}{3} \pi R^2 \cdot 8 \ m = 6 \ m^3$

$R^2 = \frac{9 \cancel{18}}{4 \cancel{8} \pi}$ meters^2

$R = \frac{3}{2\sqrt{\pi}}$ meters.

Inscribe a <u>cylinder</u> (~~inverted cone~~)

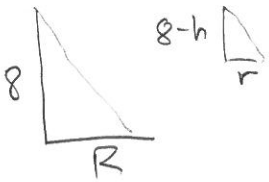
$Vol = \pi r^2 \cdot h$



(1) Find formula for volume

(2) Find r, h will yields max volume.

From picture



are similar triangles $\frac{8-h}{8} = \frac{r}{R}$ $\quad 1 - \frac{h}{8} = \frac{r}{R}$

$h = 8\left(1 - \frac{r}{R}\right)$

(1) We get $\underset{\text{"}V(r)\text{"}}{Vol} = \pi r^2 h = \pi 8 r^2 \left(1 - \frac{r}{R}\right)$ with <u>domain</u> $0 \le r \le R$.

(2) At endpoint 0, R we have $V(0) = 0$, $V(R) = 0$. There will be abs max

Abs will occur at interior critical pt.

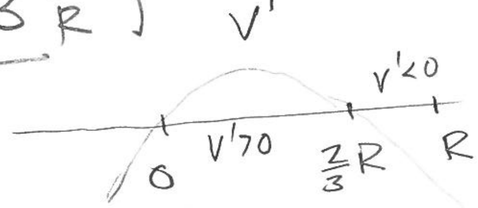
(2) $V(r) = \pi 8 \left(r^2 - \frac{r^3}{R}\right)$, so $V'(r) = \pi 8 \left(2r - 3\frac{r^2}{R}\right)$

$= \pi 8 r \left(2 - 3\frac{r}{R}\right)$.

$V' = 0$ when $\cancel{r=0}$, $r = \frac{2}{3}R$

endpt. critical pt

$V'(r) = 8\pi \underbrace{r\left(2 - 3\frac{r}{R}\right)}$

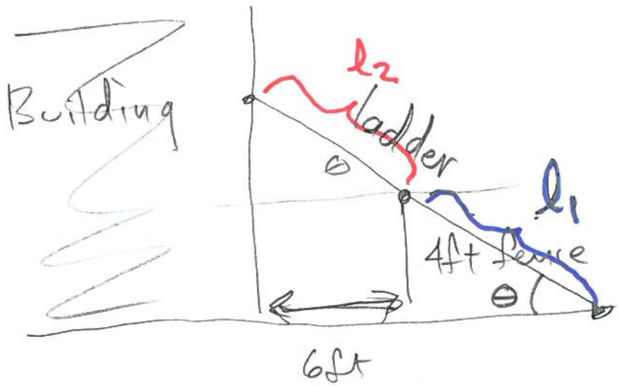


critical pt is local max

Absolute max occurs at $r = \frac{2}{3}R$

What is altitude at max? $r = \frac{2}{3}R$, $h = 8 \cdot \left(1 - \frac{r}{R}\right) = 8 \cdot \left(1 - \frac{2}{3}\right) = \frac{8}{3}$.

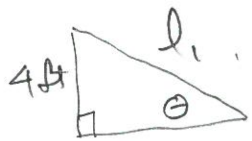
#4



ladder from ground to building

Find <u>shortest</u> ladder that can touch ground and building an clear fence.

Let θ be indicated angle.

l_1: 

$\sin\theta = \dfrac{4}{l_1}$ so $l_1 = \dfrac{4}{\sin\theta}$

l_2: 

$\cos\theta = \dfrac{6}{l_2}$ so $l_2 = \dfrac{6}{\cos\theta}$

$$\left.\right\} \; l(\theta) = \frac{4}{\sin\theta} + \frac{6}{\cos\theta}.$$

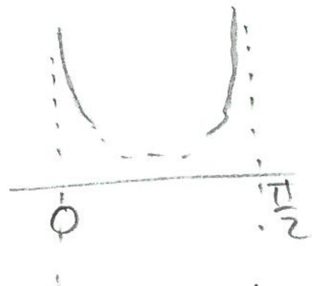
Domain of our function l is $0 < \theta < \frac{\pi}{2}$.

$\lim_{\theta \to 0^+} l(\theta) = +\infty$

$\theta = 0$ vertical asymptote.

$\lim_{\theta \to \frac{\pi}{2}^-} l(\theta) = +\infty$

$\theta = \frac{\pi}{2}$ vertical asymptote



rough graph of l.

Find critical pt of $l(\theta) = \dfrac{4}{\sin\theta} + \dfrac{6}{\cos\theta} = 4(\sin\theta)^{-1} + 6(\cos\theta)^{-1}$

$$l'(\theta) = 4\cdot(-1)(\sin\theta)^{-2}(\cos\theta) + 6(-1)(\cos\theta)^{-2}(-\sin\theta).$$

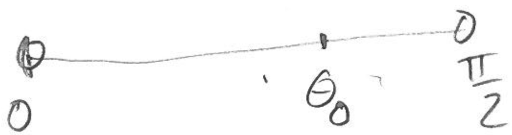
$$= -\frac{4\cos\theta}{(\sin\theta)^2} + \frac{6\sin\theta}{(\cos\theta)^2} = \frac{-4(\cos\theta)^3 + 6(\sin\theta)^3}{(\sin\theta)^2(\cos\theta)^2}$$

Since denominator alway > 0, to find critical pt we set numerator to 0

$$0 = -4(\cos\theta)^3 + 6(\sin\theta)^3$$

$$\frac{2}{3} = \left(\frac{\sin\theta}{\cos\theta}\right)^3, \text{ so } \tan\theta = \frac{\sin\theta}{\cos\theta} = \left(\frac{2}{3}\right)^{1/3}$$

$\theta_0 = \arctan\left(\frac{2}{3}\right)^{1/3}$ critical pt.



As we increase θ, $\cos\theta$ is decreasing, so $-4(\cos\theta)^3$ getting less negative
$\sin\theta$ is increasing, so $6(\sin\theta)^3$ —— more positive

This means l' flips from $-$ to $+$ at critical pt θ_0.

So $\arctan\left(\frac{2}{3}\right)^{1/3}$ is local min. Since endpts are "∞". This must be absolute min.

$$T'(x) = \frac{1}{V_{air}}\left(\frac{1}{\cancel{2}}\right)(a^2+x^2)^{-1/2}(0+\cancel{2}x) + \frac{1}{V_{glass}}\left(\frac{1}{\cancel{2}}\right)\left((1-x)^2+b^2\right)^{-1/2}\left(\cancel{2}(1-x)(-1)+0\right)$$

To find critical point

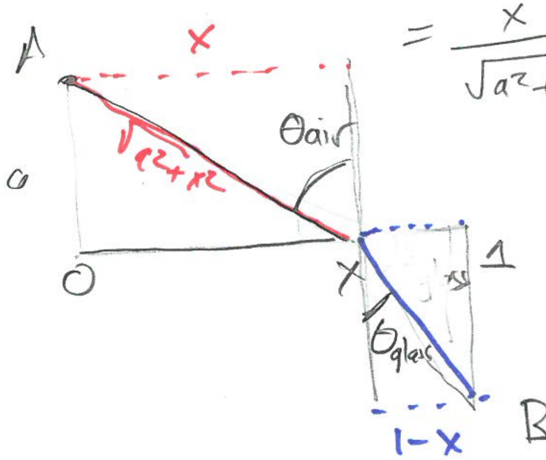
$$0 = \frac{1}{V_{air}} \cdot \frac{x}{\sqrt{a^2+x^2}} - \frac{1}{V_{glass}} \frac{(1-x)}{\sqrt{(1-x)^2+b^2}}$$

$$\frac{1}{V_{air}} \frac{x}{\sqrt{a^2+x^2}} = \frac{1}{V_{glass}} \frac{(1-x)}{\sqrt{(1-x)^2+b^2}}$$

or.

$$\frac{\left(\frac{x}{\sqrt{a^2+x^2}}\right)}{\left(\frac{(1-x)}{\sqrt{(1-x)^2+b^2}}\right)} = \frac{V_{air}}{V_{glass}} \doteq 1.5$$

$\sin\theta_{air} = \frac{x}{\sqrt{a^2+x^2}}$



so

$$\boxed{\frac{\sin\theta_{air}}{\sin\theta_{glass}} = \frac{V_{air}}{V_{glass}} = 1.5}$$

This condition is called Snell's law.

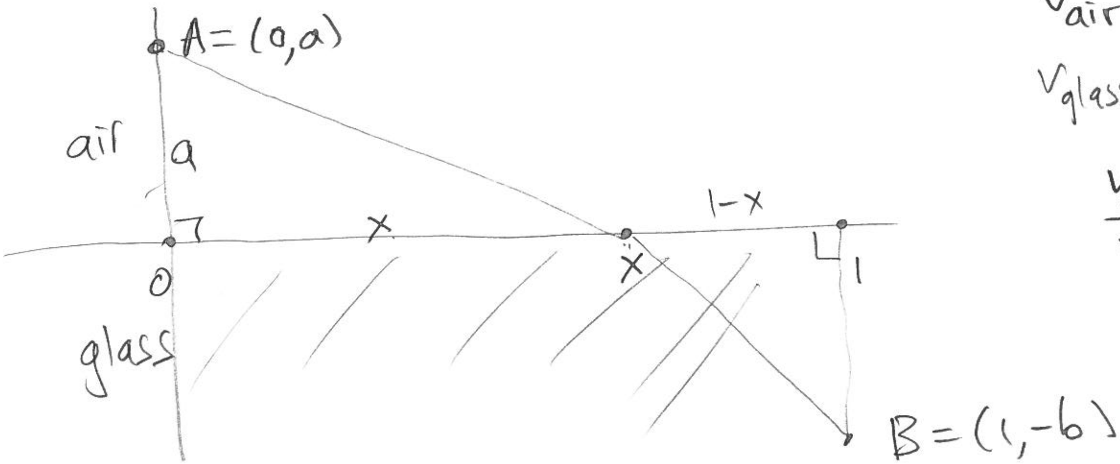
Snell's Law

Basis for why eyeglasses work.

v_{air} = speed of light in air

v_{glass} = speed of light in glass

$$\frac{v_{air}}{v_{glass}} = 1.5$$



Light travel at different speed in air and in glass.

(1) Write travel time from A to $(x,0)$ to B.

$$T(x) = \frac{\sqrt{a^2+x^2}}{v_{air}} + \frac{\sqrt{(1-x)^2+b^2}}{v_{glass}}.$$ domain $0 \le x \le 1$.

Remarkable fact. Light "finds" x so the travel time is shortest possible.

To determine this we find critical point of T.